AD NUMBER
AD275734
LIMITATION CHANGES
TO: **Approved for public release; distribution is unlimited.**
FROM: **Distribution authorized to U.S. Gov't. agencies and their contractors; Administrative/Operational Use; APR 1962. Other requests shall be referred to Office of Naval Research, One Liberty Center, 875 North Randolph Street, Arlington, VA 22203-1995.**
AUTHORITY
onr ltr 13 sep 1977

STABILITY OF EQUILIBRIUM OF CONTINUOUS BODIES

by

W. T. Koiter

Technical Report No. 79

Division of Applied Mathematics
BROWN UNIVERSITY
Providence, R. I.

April 1962

Sponsored by the Department of the Navy,
Office of Naval Research,
under contract Nonr 562(10), NR-064-406

This report contains Chapter 3 of a projected book
on the theory of elastic stability
and post-buckling behaviour.

STABILITY OF EQUILIBRIUM OF CONTINUOUS BODIES*

by

W. T. Koiter**

April 1962

---

* Chapter 3 of a projected book "Theory of Elastic Stability and Post-Buckling Behaviour".

** Visiting Professor of Applied Mathematics, Brown University, 1961-62.

Chapter 3

STABILITY OF EQUILIBRIUM OF CONTINUOUS BODIES.

3.1 The criterion of stability

3.2 Some necessary conditions for stability

3.3 A sufficient condition for stability

3.4 Reduction to a variational problem

3.5 The stability limit, neutral equilibrium and buckling modes

3.6 Stability in a critical case of neutral equilibrium

3.7 The indecisive case

3.8 The case of multiple buckling modes

3.9 Approximate solution of stability problems

References to Chapter 3

3.1 The criterion of stability.

Our discussion of the general theory of stability of equilibrium will be based on the energy criterion for conservative systems. As we have already seen in section 1.3, the implied existence of a potential energy is not too severe a restriction in the case of elastic solids or structures. An internal potential energy always exists in isothermal or adiabatic deformations of an elastic solid, and the external loads on a structure are also conservative for most static loading conditions encountered in practice.

In the present chapter (which is largely based on the analysis developed in [10, ch.2]) we shall confine our attention to an investigation of the stability of a particular configuration of equilibrium under the action of completely specified external loads. We shall denote this supposedly known equilibrium configuration as the fundamental state or state I. The energy criterion of stability of this fundamental state may then be expressed in somewhat vague general terms by the statement that the existence of a proper relative minimum of the potential energy in state I is a necessary and sufficient condition for the stability of this configuration. We shall now examine a more precise statement of this energy criterion which is appropriate for its application to continuous systems.

The investigation of the minimum properties of the potential energy in state I requires its comparison with the potential energy in all possible adjacent states II within a

certain neighbourhood of the fundamental state, to be defined presently. The possible adjacent states are characterized by the property that they satisfy the geometric conditions of support for our system. The displacement field $\underline{u}$ which carries our system from state I to a possible state II will be called kinematically admissible. The increment in potential energy of the system in the transition from state I to state II

$$P_{II}-P_{I} = P[\underline{u}] \tag{3.1.1}$$

is a functional of the kinematically admissible displacement field $\underline{u}$. It consists of a line integral, a surface integral, a volume integral or a sum of such integrals, whose integrands all depend on the local displacement vector and its spatial derivatives. We shall say that a displacement component or derivative appears explicitly in (3.1.1), if at least one of the integrands depends explicitly on this component or derivative. Let $u_i (i=1,2,3)$ denote the continuous Cartesian displacement components, and $u_{i,j}$, $u_{i,jk}$, $\ldots (i,j,k=1,2,3)$ their continuous partial derivatives with respect to Cartesian coordinates. We define a neighbourhood $\{g, g', g'', \ldots\}$ of the fundamental state I by the set of possible states II subject to the condition that the components and derivatives of the kinematically admissible displacement field from state I to state II which appear explicitly in (3.1.1) satisfy the inequalities

$$|u_i| \leqq g \ , \ |u_{i,j}| \leqq g' \ , \ |u_{i,jk}| \leqq g'' \ , \ldots, \tag{3.1.2}$$

where g, g', g",... are positive constants. A necessary and sufficient condition for the stability of equilibrium in our fundamental state I is now the existence of a non-vanishing neighbourhood $\{g, g', g'', \ldots\}$ of state I with the property that the energy increase (3.1.1) is non-negative for all possible states II in this neighbourhood. In other words,

$$P[\underline{u}] \geq 0 \qquad (3.1.3)$$

for all kinematically admissible displacement fields which satisfy (3.1.2) for some set of positive constants g, g', g",..., is a sufficient condition for stability. On the other hand, if in any neighbourhood $\{g, g', g'', \ldots\}$, no matter how small we choose the constants g,g',g",..., an admissible displacement field $\underline{u}$ exists for which $P[\underline{u}]$ is negative, then the equilibrium in the fundamental state is unstable.

The reader who is familiar with the calculus of variations will note that our formulation of the minimum property of the potential energy in a stable configuration of equilibrium is equivalent to the definition of a so-called weak minimum in the calculus of variations [e.g. 3,9]. We have two reasons for our preference for the definition of a weak minimum over the more exacting requirement of a strong minimum. The latter definition would restrict a neighbourhood of state I only by imposing bounds on the displacement components u_i themselves, and not on their derivatives. From a physical point of view, however, the boundedness of the derivatives, in so far as they appear

explicitly in the energy increase (3.1.1), is equally necessary. This physical argument is seconded by the fact that the mathematical theory of strong extrema, in particular in the case of multiple integrals, is far less well-developed than the theory of weak extrema. A detailed discussion of stability conditions on the basis of a requirement of a strong minimum of the potential energy would present nearly insurmountable difficulties in the present state of knowledge in the calculus of variations. It is indeed fortunate that the physical requirements of stability permit us to restrict our investigation to the conditions for a weak minimum.

The particular form of expression (3.1.1) for the energy increase in the transition from the fundamental state I to some possible state II is immaterial for our purposes in the present chapter. In fact, we shall deal here with some basic problems in the calculus of variations in a more general context, although we shall employ the terminology of stability theory. Nevertheless, a better understanding of the theory may be facilitated, if we mention at least some explicit examples of energy expressions in order to illustrate the otherwise somewhat abstract basic theory. In this connection it will also be convenient to anticipate in appropriate instances some results of a later detailed discussion of these examples.

As a first example we take the well-known <u>buckling problem of a simply-supported straight bar</u> of constant cross-section under the action of central compressive end loads N

(fig. 3.1). The fundamental state I

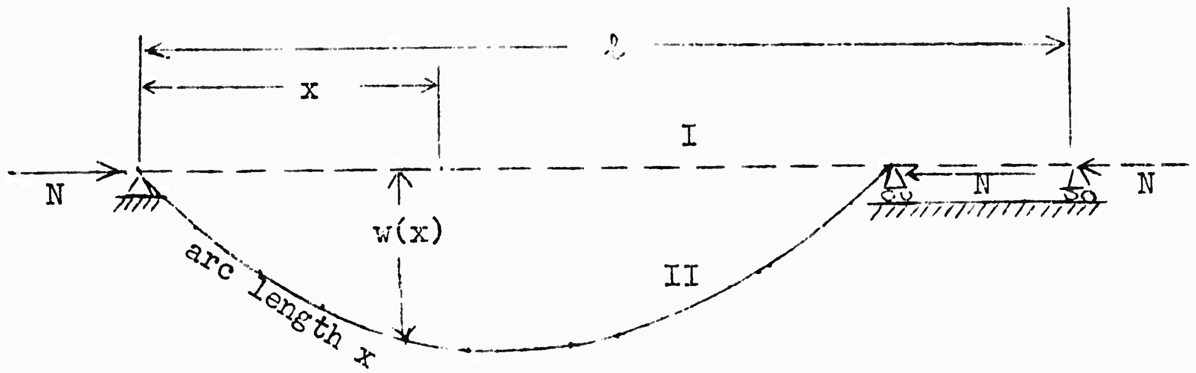

Fig. 3.1

Buckling of bar in compression

which is the subject of our investigation of stability, is the straight, undeflected configuration. We assume the center line to be inextensible. A material point on this center line is identified by its distance x to one end in state I, which distance equals the arc length in the deflected state II. We confine our attention to deflections $w(x)$ in one of the principal planes of the bar. The elastic energy is zero in state I. It amounts to $\frac{1}{2} B\kappa^2$ per unit length of the center line in state II, where κ is the curvature and B the flexural rigidity. The energy of the prescribed end loads equals the product of N and the distance between the ends of the bar. Hence we obtain in this simple case expression (3.1.1) for the energy increase in the form of a single integral

$$P_{II}-P_{I} \equiv P[w] = \int_0^{\ell} [\tfrac{1}{2} B \frac{w''^2}{1-w'^2} + N(\sqrt{1-w'^2} - 1)]dx \ , \qquad (3.1.4)$$

where primes denote differentiations with respect to x. In order that our deflection field be kinematically admissible, we must have $w(0) = w(\ell) = 0$. Equilibrium in the fundamental state I is stable, if and only if positive constants g' and g'' exist such that for all kinematically admissible deflection fields $w(x)$ which satisfy the inequalities

$$|w'| \leqq g' \ , \quad |w''| \leqq g'' \ , \qquad (3.1.5)$$

the energy increase (3.1.4) is non-negative

$$P[w] \geqq 0 \ . \qquad (3.1.6)$$

As our second example we choose the <u>three-dimensional stability problem of an elastic body</u> in a state I of initial stress described by a symmetric tensor S_{ij} $(i,j=1,2,3)$ referred to Cartesian coordinates x_i $(i=1,2,3)$. We denote the vector of body forces per unit volume in state I by $\underline{X}$, its components by X_i, and the vector of surface tractions on the part of the surface S_p where the loads are specified by $\underline{p}$, its components by p_i. We shall confine our attention to the case of <u>dead loading</u>, in which the prescribed loads on a material element of the body, specified by the vectors $\underline{X}$ dv and $\underline{p}$ dS, do not change in either magnitude or direction in the transition of the body from state I to some other possible configuration II. If the surface displacements are prescribed on a part S_u of the surface, a kinematically admissible displacement field $\underline{u}$ with components u_i must vanish

on S_u. The (additional) deformation of the body in any kinematically admissible displacement field from state I to state II is described by the strain tensor

$$\gamma_{ij} = \tfrac{1}{2}[u_{i,j}+u_{j,i}+u_{h,i}u_{h,j}] , \qquad (3.1.7)$$

where the summation convention has been employed for repeated subscripts. An appropriate expression for the increase in energy (3.1.1) will be derived in chapter 6 for small (although not infinitesimal) deformations of a solid which obeys Hooke's law. The resulting expression reads

$$P[\underline{u}] = \int_V [S_{ij}\gamma_{ij} + G\{\gamma_{ij}\gamma_{ij} + \frac{\nu}{1-2\nu}(\gamma_{hh})^2\} - X_i u_i]dv - \int_{S_p} p_i u_i dS , \qquad (3.1.8)$$

where G is the shear modulus and ν is Poisson's ratio. Equilibrium in the fundamental state I is stable, if and only if positive constants g and g' exist such that the energy increase (3.1.8) is non-negative for all kinematically admissible displacement fields $\underline{u}$ which satisfy the inequalities

$$|u_i| \leqq g \ , \ |u_{i,j}| \leqq g' \qquad (3.1.9)$$

everywhere in the body and on its surface.

3.2 Some necessary conditions for stability.

In order to apply our general energy criterion we assume that the integrands of all integrals appearing in the energy increase (3.1.1) possess in a neighbourhood (3.1.2) continuous partial derivatives with respect to all arguments u_i, $u_{i,j}$, $u_{i,jk}$ etc., which appear explicitly. For the sake of brevity we shall denote these arguments by $y^\lambda (\lambda=1,2,\ldots,q)$, and their aggregate by $\underline{y}$, where q is the number of arguments u_i, $u_{i,j}$ etc. which appear explicitly in (3.1.1). Let $F(\underline{y})$ be a typical integrand. We denote partial differentiation with respect to an argument y^λ by a subscript λ, preceded by a comma, e.g.

$$F_{,\lambda} = \frac{\partial F}{\partial y^\lambda}, \quad F_{,\lambda\mu} = \frac{\partial^2 F}{\partial y^\lambda \partial y^\mu}, \text{ etc.} \tag{3.2.1}$$

If we employ again the summation convention, now from 1 to q for a repeated Greek index, we may write the Taylor-expansion of the typical integrand in the form

$$F(\underline{y}) = y^\lambda F_{,\lambda}(0) + \frac{1}{2} y^\lambda y^\mu F_{,\lambda\mu}(0) + \ldots + \frac{1}{m!} y^{\lambda_1} \ldots y^{\lambda_m} F_{,\lambda_1 \ldots \lambda_m}(\underline{\eta}), \tag{3.2.2}$$

where the arguments η^λ of the m-th order derivatives are some positive fractions of the arguments y^λ. We collect the integrals whose integrands are homogeneous functions of their arguments of the same degree n, and we call the corresponding sum of integrals $P_n[\underline{u}]$. In this way we obtain the so-called Taylor-expansion of our energy increase functional (3.1.1) in the form

$$P[\underline{u}] = P_1[\underline{u}] + P_2[\underline{u}] + \ldots + P_m^*[\underline{u}] \ , \qquad (3.2.3)$$

where the asterisk attached to the last term indicates that it is the m-th order remainder.

It will be convenient to assume also that all geometric conditions which restrict the class of kinematically admissible displacement fields $\underline{u}$, are linear and homogeneous in the displacement components u_i and their derivatives $u_{i,j}$, $u_{i,jk}$ etc. Any linear combination of kinematically admissible displacement fields is then again such a field. In particular, we consider the linear one-parameter family of displacements*) $\alpha\underline{u}$, where $\underline{u}$ is some kinematically admissible displacement field, and α is a parameter independent of the coordinates. A necessary condition for stability is now the existence, for every kinematically admissible displacement field $\underline{u}$, of an associated positive number k such that the inequality $|\alpha| \leqq k$ ensures the inequality

$$P_1[\alpha\underline{u}] + P_2[\alpha\underline{u}] + \ldots + P_m^*[\alpha\underline{u}] =$$
$$= \alpha P_1[\underline{u}] + \alpha^2 P_2[\underline{u}] + \ldots + O(\alpha^m) \geqq 0 \ . \qquad (3.2.4)$$

---

*) A separate investigation is required in cases in which the geometric conditions are nonlinear. We may then consider a nonlinear family of admissible displacement fields $\underline{u}(\alpha)$ which reduces to zero for $\alpha=0$, but we shall not pursue the investigation of these exceptional cases.

We employ the symbol $O(\alpha^m)$ here and in the sequel for any quantity which tends to zero as α^m for $\alpha \to 0$. In the present instance it expresses the fact

$$|\alpha^{-m} P_m^*[\alpha \underline{u}]| < \infty \quad \text{for } \alpha \to 0 , \tag{3.2.5}$$

which is an immediate consequence of the boundedness of the m-th order partial derivatives of the typical integrand $F(\underline{y})$ in the neighbourhood (3.1.2).

The first necessary condition which follows from (3.2.4) is

$$P_1[\underline{u}] = 0 \tag{3.2.6}$$

for every kinematically admissible displacement field $\underline{u}$. This condition expresses the <u>principle of virtual work</u> for the equilibrium of our conservative system in its fundamental state. Since we have already assumed that state I is a configuration of equilibrium, eq.(3.2.6) does not provide any new information.

The second necessary condition for stability obtained from (3.2.4) is

$$P_2[\underline{u}] \geqq 0 . \tag{3.2.7}$$

It may be expressed in the form that the <u>second variation of the potential energy in state I must be non-negative</u> for every kinematically admissible displacement field.

If the necessary condition for stability (3.2.7) is satisfied, we have to distinguish between two cases. In the first case the equality sign in (3.2.7) holds only for an identically vanishing displacement field $\underline{u}$. The second variation

of the energy is then positive for every non-vanishing admissible displacement field, and this second variation will be called <u>positive definite</u>. In this case we cannot infer from (3.2.4) any further necessary conditions for stability. The second case arises, if the second variation is again always non-negative but if it takes the value zero for some non-vanishing displacement field, say $\underline{u}_1$. The second variation is called <u>positive semi-definite</u> in this case, and we obtain two further necessary conditions for stability from (3.2.4) in the form

$$P_3[\underline{u}_1] = 0 \; , \tag{3.2.8}$$

$$P_4[\underline{u}_1] \geqq 0 \; . \tag{3.2.9}$$

Equation (3.2.8) and inequality (3.2.9) must hold for every kinematically admissible displacement field $\underline{u}_1$ for which the second variation vanishes.

If the additional conditions (3.2.8) and (3.2.9) in the case of a semi-definite second variation are both satisfied, we have again to distinguish between the case in which the fourth variation $P_4[\underline{u}_1]$ is positive definite (i.e. positive for all non-vanishing displacement fields $\underline{u}_1$ for which $P_2[\underline{u}_1] = 0$), and the case in which the fourth variation is semi-definite. In the first case we obtain no new necessary conditions from (3.2.4), in the second case we infer additional necessary conditions for the fifth and sixth variations, similar to (3.2.8) and (3.2.9), etc.

We emphasize again that our discussion up to now has resulted only in a number of necessary conditions for stability. The question to what extent these conditions are perhaps also sufficient conditions for stability is still completely open at this stage; this question will be discussed in detail in the next sections. In the conventional treatment of stability theory on the basis of the energy criterion the discussion is usually not pursued beyond (3.2.7). It is then taken for granted that a positive definite second variation of the energy is also a sufficient condition for stability. This complacent view is presumably based on a tacit assumption that (3.2.4) also supplies sufficient conditions for stability. This tacit assumption, however, is gravely in error. We shall see later, in section 3.6, that equilibrium in the fundamental state may very well be unstable, in spite of the satisfaction of (3.2.4) for every kinematically admissible displacement field. For example, even in the case of a positive definite fourth variation $P_4[\underline{u}_1]$, (3.2.8) and (3.2.9) are not sufficient conditions for stability.

We conclude this section with some explicit formulae for the various terms in the Taylor-expansion of the energy expressions for our examples discussed earlier. In the case of buckling of a compressed bar we have from (3.1.4)

$$P_1[w] \equiv 0 \ , \tag{3.2.10}$$

$$P_2[w] = \int_0^{\ell} [\tfrac{1}{2} Bw''^2 - \tfrac{1}{2} Nw'^2]dx \ , \tag{3.2.11}$$

$$P_3[w] \equiv 0 \ , \tag{3.2.12}$$

$$P_4[w] = \int_0^{\ell} [\tfrac{1}{2} Bw'^2 w''^2 - \tfrac{1}{8} Nw'^4]dx \ , \tag{3.2.13}$$

$$P_5[w] \equiv 0 \ , \tag{3.2.14}$$

$$P_6[w] = \int_0^{\ell} [\tfrac{1}{2} Bw'^4 w''^2 - \tfrac{1}{16} Nw'^6]dx \ . \tag{3.2.15}$$

<u>For the stability problem of a three-dimensional elastic body under dead loading</u> we have from (3.1.7) and (3.1.8)

$$P_1[\underline{u}] = \int_V [\tfrac{1}{2} S_{ij}(u_{i,j}+u_{j,i}) - X_i u_i]dv - \int_{S_p} p_i u_i dS \ , \tag{3.2.16}$$

$$P_2[\underline{u}] = \int_V [\tfrac{1}{2} S_{ij}u_{h,i}u_{h,j}+G\{\tfrac{1}{4}(u_{i,j}+u_{j,i})(u_{i,j}+u_{j,i}) + \frac{\nu}{1-2\nu}(u_{h,h})^2\}]dv \ , \tag{3.2.17}$$

$$P_3[\underline{u}] = \int_V G[\tfrac{1}{2}(u_{i,j}+u_{j,i})u_{h,i}u_{h,j} + \frac{\nu}{1-2\nu} u_{k,k}u_{h,i}u_{h,i}]dv \ , \tag{3.2.18}$$

$$P_4[\underline{u}] = \int_V G[\tfrac{1}{4} u_{h,i}u_{h,j}u_{k,i}u_{k,j} + \tfrac{1}{4}\frac{\nu}{1-2\nu}(u_{h,i}u_{h,i})^2]dv \ . \tag{3.2.19}$$

The first variation of the energy must of course vanish in both examples, because the fundamental state is a configuration of equilibrium. In our first example this fact is immediately confirmed by (3.2.10). In our second example we have to keep in mind that the distribution of initial

stresses S_{ij} must satisfy the equations of equilibrium

$$S_{ij,j} + X_i = 0 \tag{3.2.20}$$

in the interior of the body, and the boundary conditions

$$S_{ij}n_j = p_i \tag{3.2.21}$$

on the surface portion S_p. The vanishing of the first variation (3.2.16) is now easily confirmed by means of the divergence theorem (cf. section 2.4 of the appendix) and the geometric conditions $u_i = 0$ on S_u. Finally, we note that the Taylor-expansion of the energy expression (3.1.8) for the second example <u>terminates</u> with its fourth variation (3.2.19).

3.3 A sufficient condition for stability.

It will first be shown, by means of a simple counter-example of a well-known type in the calculus of variations [e.g. 3,9], that a positive definite second variation is not always a sufficient condition for a proper relative minimum. Our counter-example is admittedly somewhat artificial from a physical point of view. Nevertheless it indicates clearly why the criterion of a positive definite second variation may occasionally fail as a sufficient condition. It also points the way to a slightly modified and uniformly valid sufficient condition for stability, based on the second variation. Our modified criterion reduces to the condition of a positive definite second variation in all conceivable practical applications (cf. section 3.4).

In our counter-example we consider an expression for the energy increase (3.1.1) in the form of a single integral

$$P[u] = \int_0^1 (x^2u'^2-u'^3)dx \,, \qquad (3.3.1)$$

where $u(x)$ is the single displacement component, and primes denote differentiations with respect to x. The geometric conditions for a kinematically admissible displacement field are assumed to be $u(0)=u(1)=0$.

The first variation of (3.3.1) is identically zero, and the second variation

$$P_2[u] = \int_0^1 x^2u'^2dx \qquad (3.3.2)$$

is evidently positive definite. We now consider a one-parametric family of displacement fields $u(x;\varepsilon)$

$$\left.\begin{aligned} &0 \leqq x \leqq \varepsilon < 1 \;:\; u = \varepsilon[x-2x^2/\varepsilon+x^3/\varepsilon^2] \;, \\ &\varepsilon \leqq x \leqq 1 \qquad :\; u = 0 \;. \end{aligned}\right\} \tag{3.3.3}$$

Both the displacement and its first derivative are continuous and tend to zero for $\varepsilon \to 0$. The energy increase (3.3.1) for the displacement field $u(x;\varepsilon)$ is easily evaluated. The result is

$$P[u(x;\varepsilon)] = \frac{2}{105}\,\varepsilon^5 - \frac{2}{35}\,\varepsilon^4 \;, \tag{3.3.4}$$

where the first term is due to the second variation. This result is obviously negative for all ε in the range $0 < \varepsilon \leqq 1$, and it follows that the energy increase, and therefore the energy itself, has no proper minimum for $u \equiv 0$.

The failure of the positive definite second variation as a criterion of stability in the present example is of course due to the factor x^2 in the integrand of (3.3.2). No matter how narrow we choose the neighbourhood of $u \equiv 0$, that is no matter how small we choose the positive bound g' for the modulus of u', there always exist displacement fields for which the cubic term in (3.3.1) is the dominant term.

An obvious way out of our difficulty would be to require that not only the second variation itself but also its <u>integrand</u> is positive definite. Thus it might be ensured that the third-order remainder is always less in magnitude than the

second variation for a sufficiently narrow neighbourhood of the fundamental state. However, our mark would be overshot by a wide margin by this more sweeping criterion. Few problems of elastic stability where equilibrium is still stable, would meet the too exacting sufficient condition of a positive definite integrand of the second variation of the energy. This condition is certainly sufficient, but it is by no means necessary.

Our purpose is actually achieved not by requiring that the second variation should have a positive definite integrand, but by comparing $P_2[\underline{u}]$ with an auxiliary homogeneous quadratic functional $T_2[\underline{u}]$ whose typical integrand is a positive definite quadratic function of all arguments which appear explicitly in the energy increase (3.1.1).

It is convenient to rewrite the typical integrand of the second variation in the form*)

$$\frac{1}{2} F_{,\lambda\mu}(0) y^\lambda y^\mu = [\frac{1}{2} F_{,\lambda\mu}(0) + C_{\lambda\mu}] y^\lambda y^\mu - C_{\lambda\mu} y^\lambda y^\mu \; , \qquad (3.3.5)$$

where the constants $C_{\lambda\mu}$ are chosen in such a way that both quadratic forms in (3.3.5) are positive definite in all arguments which appear explicitly in $F(\underline{y})$. Without loss in generality we may take $C_{\lambda\mu} = 0$ for $\lambda \neq \mu$. The non-vanishing constants $C_{\lambda\lambda}$ (λ not summed) are then all positive. The sum of integrals whose integrands are given by $C_{\lambda\mu} y^\lambda y^\mu$ is denoted as the auxiliary quadratic functional $T_2[\underline{u}]$. Since both terms in (3.3.5) are

---

*) The basic idea of splitting up the typical integrand in the form (3.3.5) seems to be due to Trefftz [18], although he does not pursue the argument to its end.

positive, we have the inequality

$$\frac{P_2[\underline{u}]}{T_2[\underline{u}]} > -1. \tag{3.3.6}$$

This inequality ensures the existence of a greatest lower bound (g.ℓ.b.), say d, for the left-hand member of (3.3.6). This greatest lower bound is characterized by two properties. It ensures the inequality

$$\frac{P_2[\underline{u}]}{T_2[\underline{u}]} \geqq d \tag{3.3.7}$$

for every non-vanishing kinematically admissible displacement field $\underline{u}$. On the other hand, it implies for every positive number e, the existence of a kinematically admissible displacement field $\underline{v}$ satisfying the inequality

$$\frac{P_2[\underline{v}]}{T_2[\underline{v}]} < d+e \ . \tag{3.3.8}$$

We shall now prove that a positive value of the greatest lower bound, $d > 0$, is a sufficient condition for stability. From our necessary condition (3.2.7), and from (3.3.8) we have already a necessary condition for stability in terms of this lower bound, viz. $d \geqq 0$. In the case of a semi-definite second variation we must have $d=0$. We know from our previous discussion that the additional necessary conditions (3.2.8) and (3.2.9) have to be satisfied for stability in that case. It follows that our criterion $d > 0$ is the sharpest possible sufficient condition for stability in terms of the second variation alone.

The actual proof of our sufficient condition for stability is comparatively simple. From (3.2.3) and (3.2.6) we have

$$P[\underline{u}] = P_2[\underline{u}] + P_3^*[\underline{u}] \ . \tag{3.3.9}$$

Consulting (3.2.2), we write the typical integrand of the third-order remainder in the form

$$\frac{1}{6}\, y^\lambda y^\mu y^\nu F_{,\lambda\mu\nu}(\underline{\eta}) = [\tfrac{1}{6}\, y^\nu F_{,\lambda\mu\nu}(\underline{\eta})]y^\lambda y^\mu = A_{\lambda\mu} y^\lambda y^\mu \ , \tag{3.3.10}$$

where $A_{\lambda\mu}$ is an abbreviation of the expression between brackets. Remembering our particular choice for the integrands of $T_2[\underline{u}]$, we may write

$$|y^\lambda| \leqq [\frac{1}{C_{\lambda\lambda}}\, C_{\alpha\beta} y^\alpha y^\beta]^{\frac{1}{2}} \quad (\lambda \text{ not summed}), \tag{3.3.11}$$

Introducing the new abbreviation

$$\frac{1}{\sqrt{C_{\lambda\lambda}}} = c^\lambda \quad (\lambda \text{ not summed}), \tag{3.3.12}$$

we obtain the inequality

$$|A_{\lambda\mu} y^\lambda y^\mu| \leqq |A_{\lambda\mu}| c^\lambda c^\mu \cdot C_{\alpha\beta} y^\alpha y^\beta \ . \tag{3.3.13}$$

We now recall that the third-order derivatives of the typical integrand are bounded. Hence we may make the quantities $A_{\lambda\mu}$ in (3.3.10) and (3.3.13) as small in modulus as we please by restricting our attention to a sufficiently close neighbourhood of the fundamental state, i.e. by choosing the positive constants g,g',g", ... in (3.1.2) sufficiently small. It follows that the

typical integrand of the third-order remainder may be reduced in absolute value to an arbitrarily small positive fraction of the typical integrand of $T_2[\underline{u}]$. The same conclusion then applies a fortiori to the corresponding integrals or sums of integrals. We have therefore by a suitable choice of the neighbourhood $\{g, g', g'', \ldots\}$

$$|P_3^*[\underline{u}]| \leqq \varepsilon\, T_2[\underline{u}] \,, \tag{3.3.14}$$

where ε is a positive number as small as we please. From (3.3.7), (3.3.9) and (3.3.14) we have finally

$$P[\underline{u}] \geqq (d-\varepsilon) T_2[\underline{u}] \,, \tag{3.3.15}$$

and the sufficiency for stability of the criterion $d > 0$ has been proven.

It will be observed that the particular choice for the typical integrand of $T_2[\underline{u}]$ is immaterial, provided that it is positive definite in all arguments appearing in the energy increase functional (3.1.1). We might even relax this requirement slightly. It is indeed sufficient that the typical integrand of $T_2[\underline{u}]$ is positive definite in all arguments which appear explicitly in the third-order remainder (cf. (3.3.10)). The independence of our criterion of the particular choice of typical integrand for $T_2[\underline{u}]$ results from the inequalities, holding for any pair of positive definite quadratic forms [5, ch.1]

$$C^*_{\lambda\mu} y^\lambda y^\mu \leqq C^* \cdot C_{\lambda\mu} y^\lambda y^\mu \,, \tag{3.3.16}$$

$$C^*_{\lambda\mu} y^\lambda y^\mu \geqq c^* \cdot C_{\lambda\mu} y^\lambda y^\mu \,, \tag{3.3.17}$$

where C^* and c^* are suitable positive numbers. The actual value of the greatest lower bound d, defined by (3.3.7) and (3.3.8), does of course depend on the choice of $T_2[\underline{u}]$, but a positive value of d corresponding to $T_2[\underline{u}]$ implies a positive value d* corresponding to $T_2^*[\underline{u}]$, and d=0 implies $d^*=0$, and vice versa. We shall see in the next section that a much larger freedom in our choice of $T_2[\underline{u}]$ even exists for most practical applications.

Finally, it may be worthwhile to return briefly to our counter-example (3.3.1) concerning the criterion of a positive definite second variation of the energy as a sufficient condition for stability. It is easily verified that our modified criterion based on the second variation is not in contradiction to the non-existence of a minimum in this example. If we take

$$T_2[u] = \int_0^1 u'^2 dx \, , \tag{3.3.18}$$

we obtain by means of the one-parametric family of displacement fields (3.3.3)

$$d = \text{g.}\ell\text{.b.} \frac{P_2[u]}{T_2[u]} = \text{g.}\ell\text{.b.} \frac{\int_0^1 x^2 u'^2 dx}{\int_0^1 u'^2 dx} \leqq \lim_{\varepsilon \to 0} \frac{1}{7}\varepsilon^2 = 0 \, . \tag{3.3.19}$$

Since we know from the positive definite character of the second variation (3.3.2) that the greatest lower bound d is non-negative, it must necessarily be zero. Our modified sufficient condition for stability thus fails to yield a decision, and no contradiction arises between our criterion and the previously established non-existence of a minimum of (3.3.1) for $u \equiv 0$.

3.4 Reduction to a variational problem.

Before we continue our discussion it may be helpful to summarize the results of the analysis in the preceding section. Let $T_2[\underline{u}]$ denote a homogeneous quadratic functional with positive definite integrands in all arguments which appear explicitly in the energy increase (3.1.1). Let d denote the greatest lower bound of the quotient of the second variation of the energy $P_2[\underline{u}]$ and $T_2[\underline{u}]$

$$d = g.\ell.b. \frac{P_2[\underline{u}]}{T_2[\underline{u}]} . \tag{3.4.1}$$

A necessary condition for stability of our fundamental state is then $d \geqq 0$, a sufficient condition for stability is $d > 0$.

Remarkably simple as this criterion may appear, it is very difficult to apply unless we can assume that the greatest lower bound d is actually attained for some non-vanishing kinematically admissible displacement field. This assumption implies the <u>existence of a minimum of the quotient of the two functionals</u> $P_2[\underline{u}]$ and $T_2[\underline{u}]$. We emphasize that the existence of such a minimum cannot be proved in complete generality. On the contrary, counter-examples are easily constructed in which no minimum exists, in spite of the existence of a greatest lower bound. A simple example of this type is provided by the quotient of functionals in (3.3.19). The greatest lower bound zero in this case is not attained for any non-vanishing kinematically admissible function u(x). The non-existence of a minimum is again due to the factor x^2 in the integrand of the second variation. No

similar difficulty would occur, if this factor were replaced by a continuously differentiable function of x with a positive minimum in the closed interval $0 \leqq x \leqq 1$.

In order to come to grips with our stability problem, we shall henceforward assume that the minimum problem

$$\omega_1 = \text{Min.} \frac{P_2[\underline{u}]}{T_2[\underline{u}]} \tag{3.4.2}$$

actually has a solution, specified by some non-vanishing kinematically admissible displacement field $\underline{u}_1$. We are guided in this assumption by our physical intuition that the actual structure of the integrands of the second variation $P_2[\underline{u}]$ in a physically meaningful stability problem will ensure the existence of a solution. We derive some support for our intuition from the theory of small vibrations mentioned below. It must be admitted, however, that a complete theory would require a verification, at least a posteriori, that a solution of our minimum problem (3.4.2) actually exists. Such an existence proof is indeed available for a wide class of problems of type (3.4.2), but we shall not pursue this aspect of the theory. The interested reader is referred to the treatise by Courant and Hilbert [6] for a detailed discussion of existence theory in variational problems of quadratic functionals.

We may now rephrase our previous criterion of stability in terms of the solution ω_1 of our minimum problem (3.4.2): a necessary condition for stability of our fundamental state I is $\omega_1 \geqq 0$, a sufficient condition for stability is $\omega_1 > 0$. No

decision is obtained as yet in the critical case $\omega_1 = 0$, which case will be discussed in the next sections.

An interesting consequence of our assumption of the existence of a solution to the minimum problem (3.4.2) is that a positive definite second variation of the energy is now actually a sufficient condition for stability. For if a solution to problem (3.4.2) exists, it is necessarily positive if the second variation is positive definite. Hence we return to the criterion of stability in its customary form, which we had discredited in section 3.2. A casual reader may now well question the purpose of our entire discussion in section 3.3. We underline therefore that our preceding discussion has put the conventional criterion of a positive definite second variation as a sufficient condition for stability in its proper perspective. It is justified, if and only if a solution of the minimum problem (3.4.2) exists.

A second important consequence of our assumption of a solution to (3.4.2) is that we may now replace the functional $T_2[\underline{u}]$, whose integrand is positive definite in all arguments appearing explicitly in (3.1.1), by any positive definite homogenerous quadratic functional $\overset{*}{T}_2[\underline{u}]$, and consider the minimum problem

$$\overset{*}{\omega}_1 = \text{Min.}\ \frac{P_2[\underline{u}]}{\overset{*}{T}_2[\underline{u}]}\ . \tag{3.4.3}$$

If a solution to (3.4.3) also exists, it must obviously have the same sign as the solution of (3.4.2), and if one of the minima

ω_1 or ω_1^* is zero, the other one must also be zero. The corresponding displacement fields $\underline{u}_1$ and $\underline{u}_1^*$, for which the minima ω_1 and ω_1^* are attained, are of course not the same in general, but if $\omega_1 = \omega_1^* = 0$, we may also identify the displacement fields $\underline{u}_1$ and $\underline{u}_1^*$. The increased freedom in the choice of an auxiliary quadratic functional $T_2[\underline{u}]$ is convenient for many applications. Whenever we use this greater freedom, we shall mark this functional by an asterisk as in (3.4.3). Our freedom is of course essentially limited by the requirement that the functional $T_2^*[\underline{u}]$ must be positive definite, even if its typical integrand need not be definite.

As a first illustration of the wider class of functionals $T_2^*[\underline{u}]$, we consider the expression for the kinetic energy of our mechanical system. Replacing the velocities by the displacement components themselves, we obtain a functional of type $T_2^*[\underline{u}]$, whose typical integrand is a positive definite quadratic function of the displacement components. The minimum property (3.4.3) expresses in this case Rayleigh's principle for the square of the fundamental frequency of free (small) vibrations of our system. The displacement field for which the minimum is attained, is the corresponding fundamental mode. The existence of such a fundamental mode of vibration is a well-established experimental fact. It ensures the existence of a solution to the minimum problem (3.4.3) in this case, and it supports the assumed existence of a solution to our original minimum problem (3.4.2).

We now return to our basic minimum problem (3.4.2). Let $\underline{u}_1$ again denote the displacement field for which the minimum ω_1 is attained. Let $\underline{\zeta}$ denote an arbitrary kinematically admissible displacement field. Since our geometric conditions are linear and homogeneous, the displacement field $\underline{u}_1+\varepsilon\underline{\zeta}$ is also kinematically admissible for any value of the constant ε. The minimum property of the solution of (3.4.2) is then expressed by

$$\frac{P_2[\underline{u}_1+\varepsilon\underline{\zeta}]}{T_2[\underline{u}_1+\varepsilon\underline{\zeta}]} \geqq \frac{P_2[\underline{u}_1]}{T_2[\underline{u}_1]} = \omega_1 , \qquad (3.4.4)$$

whence

$$P_2[\underline{u}_1+\varepsilon\underline{\zeta}] - \omega_1 T_2[\underline{u}_1+\varepsilon\underline{\zeta}] \geqq 0 . \qquad (3.4.5)$$

We introduce a convenient notation for the binomial expansion of any functional $S_m[\underline{u}]$, whose typical integrand is a homogeneous polynomial of degree m in the components of the displacement u_i and their derivatives $u_{i,j}$, $u_{i,jk}$, etc. We write, if $\underline{u} = \underline{v}+\underline{w}$,

$$S_m[\underline{v}+\underline{w}] = \sum_{n=o}^{m} S_{(m-n)n}[\underline{v},\underline{w}] , \qquad (3.4.6)$$

where the typical integrand of $S_{(m-n)n}[\underline{v},\underline{w}]$ contains all terms in the binomial expansion of degree (m-n) in $\underline{v}$ and its derivatives, and of degree n in $\underline{w}$ and its derivatives. Accordingly we have the identities

$$S_{(m-n)n}[\underline{v},\underline{w}] = S_{n(m-n)}[\underline{w},\underline{v}], \quad S_{(m-n)n}[\underline{v},\underline{v}] = \frac{m!}{(m-n)!n!} S_m[\underline{v}] ,$$

$$S_{mo}[\underline{v},\underline{w}] = S_{om}[\underline{w},\underline{v}] = S_m[\underline{v}] . \qquad (3.4.7)$$

Applying the binomial expansion (3.4.6) to both terms in (3.4.5), and remembering the identity

$$P_2[\underline{u}_1] = \omega_1 T_2[\underline{u}_1] \ , \tag{3.4.8}$$

we obtain from (3.4.5)

$$\varepsilon\{P_{11}[\underline{u}_1,\underline{\zeta}] - \omega_1 T_{11}[\underline{u}_1,\underline{\zeta}]\} + \varepsilon^2\{P_2[\underline{\zeta}] - \omega_1 T_2[\underline{\zeta}]\} \geqq 0 \ . \tag{3.4.9}$$

This inequality must hold for arbitrary values of the constant ε. A necessary condition for (3.4.9) to hold is therefore

$$P_{11}[u_1,\underline{\zeta}] - \omega_1 T_{11}[\underline{u}_1,\underline{\zeta}] = 0 \tag{3.4.10}$$

for every kinematically admissible displacement field $\underline{\zeta}$. In view of the minimum property of ω_1 (3.4.2), this condition is also sufficient to satisfy (3.4.9), because the second term is obviously always non-negative.

It is convenient to formulate our resulting equation (3.4.10) in the slightly different form that $\omega=\omega_1$ and $\underline{u}=\underline{u}_1$ are a solution of the equation

$$P_{11}[\underline{u},\underline{\zeta}] - \omega T_{11}[\underline{u},\underline{\zeta}] = 0 \ , \tag{3.4.11}$$

holding for every kinematically admissible displacement field $\underline{\zeta}$. We shall call this more general equation the variational equation associated with the quotient of our two quadratic functionals $P_2[\underline{u}]$ and $T_2[\underline{u}]$. If $\underline{u}$ is a non-vanishing solution of (3.4.11) for some value of the parameter ω, we obtain from the equation

itself, evaluated for $\underline{\zeta} = \underline{u}$,

$$\omega = \frac{P_2[\underline{u}]}{T_2[\underline{u}]} , \tag{3.4.12}$$

where appropriate use has been made of (3.4.7). An even more significant property of a solution $\underline{u}$ of (3.4.11) is that it yields a stationary value of the quotient in (3.4.12). This statement is based on the easily verified fact that the quotient in (3.4.12), evaluated for any kinematically admissible displacement field $\underline{u}+\varepsilon\underline{\zeta}$, differs from (3.4.12) by an amount which is quadratic in ε, if $\underline{u}$ is a kinematically admissible non-vanishing solution of (3.4.11). The absence of a linear term in ε is characteristic of the stationary property of ω in (3.4.12).

We note that the variational equation is homogeneous and linear in the displacement field $\underline{u}$. Hence it will always have a trivial solution $\underline{u} \equiv 0$. Non-vanishing kinematically admissible solutions of (3.4.11) will be called (vector) eigensolutions. As a rule, they exist only for special values of the parameter ω, the so-called eigenvalues which coincide with the stationary values (3.4.12). In view of the homogeneity of eq. (3.4.11) any eigensolution must contain an indeterminate constant factor. In order to remove this ambiguity we shall normalise our eigensolutions by requiring that the functional $T_2[\underline{u}]$ takes a fixed positive value. It is usually expedient to normalise by taking $T_2[\underline{u}]$ equal to unity, and we shall also adopt this convention. The normalised eigensolutions will be called (vector) eigenfunctions. We shall order our eigenvalues in an increasing

sequence $\omega_1 \leqq \omega_2 \leqq \omega_3 \leqq \ldots.$, and the associated eigenfunctions will be called $\underline{u}_1$, $\underline{u}_2$, $\underline{u}_3$, ... This nomenclature*) agrees of course with our previous definition of ω_1 and $\underline{u}_1$, although we have now disposed of a previously indeterminate constant factor in $\underline{u}_1$. For each combination of eigenvalue and associated eigenfunction we have the identity

$$\omega_j = \frac{P_2[\underline{u}_j]}{T_2[\underline{u}_j]} . \tag{3.4.13}$$

An important property of the eigenfunctions $\underline{u}_h$ and $\underline{u}_k$, associated with different eigenvalues ω_h and ω_k, is that they are orthogonal with reference to the quadratic functional $T_2[\underline{u}]$. Orthogonality of two kinematically admissible displacement fields $\underline{v}$ and $\underline{w}$ is here defined by the relation

$$T_{11}[\underline{v},\underline{w}] = 0 . \tag{3.4.14}$$

We note that the positive definite character of $T_2[\underline{u}]$ implies linear independence of the displacement fields $\underline{v}$ and $\underline{w}$ in (3.4.14). The proof of our statement with respect to the orthogonality of $\underline{u}_h$ and $\underline{u}_k$ follows immediately from our variational equations (3.4.11), satisfied by $\underline{u}_h$ for $\omega=\omega_h$, and by $\underline{u}_k$ for $\omega=\omega_k$. Taking $\underline{\zeta}=\underline{u}_k$ in the first equation, $\underline{\zeta}=\underline{u}_h$ in the second equation, and subtracting, we obtain by means of (3.4.7)

$$(\omega_h-\omega_k)T_{11}[\underline{u}_h,\underline{u}_k] = 0 . \tag{3.4.15}$$

*) The reader should distinguish carefully between our present notation for the eigenfunctions $\underline{u}_h$ $(h=1,2,\ldots)$, each of which represents a vector field with local displacement components u_{h1}, u_{h2}, u_{h3}, and our previous notation $u_j (j=1,2,3)$ for the displacement components themselves.

Since $\omega_h \neq \omega_k$, we have

$$T_{11}[\underline{u}_h, \underline{u}_k] = 0 \ . \qquad (3.4.16)$$

By means of the variational equation for $\underline{u}_h$ or $\underline{u}_k$, we obtain also

$$P_{11}[\underline{u}_h, \underline{u}_k] = 0 \ . \qquad (3.4.17)$$

Our proof of orthogonality of two eigenfunctions $\underline{u}_h$ and $\underline{u}_k$ breaks down, if the associated eigenvalues are equal, $\omega_h = \omega_k$. The general case of a multiple eigenvalue is described by

$$\omega_h = \omega_{(h+1)} = \dots = \omega_{(h+n)} \ . \qquad (3.4.18)$$

The equality of these (n+1) eigenvalues implies that the variational equation has (n+1) linearly independent eigensolutions for this particular value of ω. Let $\underline{v}_h$, $\underline{v}_{(h+1)}, \dots, \underline{v}_{(h+n)}$ denote the associated normalised eigensolutions or eigenfunctions. From these we may now construct a new set of (n+1) eigenfunctions which satisfy our orthogonality condition (3.4.16). The relation (3.4.17) follows then again immediately from the variational equation.

By linear combination of our available eigenfunctions $\underline{v}_h, \dots, \underline{v}_{(h+n)}$ we construct eigensolutions defined by the recurrent scheme

$$\underline{w}_{(h+k)} = \underline{v}_{(h+k)} + \sum_{j=o}^{k-1} \alpha_{kj} \, \underline{w}_{(h+j)} \ , \ k=0,1,2,\dots n \qquad (3.4.19)$$

where α_{kj} $(k=1,2,\dots n; j < k)$ are constant coefficients, or in more explicit form

$$\underline{w}_h = \underline{v}_h$$

$$\underline{w}_{(h+1)} = \underline{v}_{(h+1)} + \alpha_{10}\,\underline{w}_h$$

$$\underline{w}_{(h+2)} = \underline{v}_{(h+2)} + \alpha_{20}\,\underline{w}_h + \alpha_{21}\,\underline{w}_{(h+1)}\,,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\underline{w}_{(h+n)} = \underline{v}_{(h+n)} + \alpha_{no}\,\underline{w}_h + \cdots + \alpha_{n(n-1)}\underline{w}_{(h+n-1)}\,.$$

From the construction (3.4.19) it is immediately obvious that our new eigensolutions are again linearly independent. We determine the coefficients α_{kj} ($k=1,2,\ldots n; j < k$) recurrently from the requirement that each eigensolution $\underline{w}_{(h+k)}$ shall be orthogonal to all its predecessors $\underline{w}_{(h+j)}$, $j < k$. We obtain from this condition

$$\alpha_{kj} = -\frac{T_{11}[\underline{v}_{(h+k)}, \underline{w}_{(h+j)}]}{2T_2[\underline{w}_{(h+j)}]}\,. \qquad (3.4.20)$$

Normalising the (n+1) orthogonal eigensolutions $\underline{w}_{(h+j)}$ ($j=0,1,2,\ldots,n$), we have indeed constructed a set of (n+1) mutually orthogonal eigenfunctions associated with the multiple eigenvalue (3.4.18).

Henceforward we shall assume that we have already orthogonalised the eigenfunctions associated with a multiple eigenvalue. Orthogonality is then a general property of any pair of eigenfunctions which enter our discussion.

An important consequence of the orthogonality of eigenfunctions associated with different eigenvalues is that all eigenvalues are real. This theorem is easily proved indirectly. If it were false, and a complex eigenvalue $\lambda = +i\mu$

would exist, its associated eigenfunction would also possibly be complex $\underline{u} = \underline{v}+i\underline{w}$. Its complex conjugate $\underline{v}-i\underline{w}$ would then also satisfy the variational equation (3.4.11) for the complex conjugate $\lambda-i\mu$ of ω. The orthogonality relation

$$T_{11}[(\underline{v}+i\underline{w}), (\underline{v}-i\underline{w})] = 0 \qquad (3.4.21)$$

is incompatible with the expansion of its left-hand member

$$T_{11}[(\underline{v}+i\underline{w})(\underline{v}-i\underline{w})] = 2T_2[\underline{v}] + 2T_2[\underline{w}] > 0 . \qquad (3.4.22)$$

Our contradiction between (3.4.21) and (3.4.22) implies that the assumption of existence of a complex eigenvalue is untenable.

We note that eigensolutions of the variational equation are not necessarily real, but the real and imaginary parts of a complex eigensolution, taken separately, must each satisfy this equation. Any complex eigensolution may therefore be represented as a linear combination of real eigenfunctions with complex coefficients. Without loss in generality we may restrict our attention to the real eigenfunctions.

A further important consequence of our orthogonality relations is that we may write any kinematically admissible displacement field $\underline{u}$ as the sum of a linear combination of a number of eigenfunctions and a residual displacement field $\underline{v}$ orthogonal to these eigenfunctions, for instance

$$\underline{u} = \sum_{j=1}^{k} a_j\underline{u}_j + \underline{v}, \text{ where } T_{11}[\underline{u}_j,\underline{v}] = 0, \; j=1,2,\ldots k. \qquad (3.4.23)$$

In order to prove this thoerem we write the required orthogonality relations for $\underline{v}$ in the form

$$T_{11}[\underline{u}_h, \{\underline{u} - \sum_{j=1}^{k} a_j\underline{u}_j\}] = T_{11}[\underline{u}_h,\underline{u}] - \sum_{j=1}^{k} a_j T_{11}[\underline{u}_h,\underline{u}_j] =$$

$$= T_{11}[\underline{u}_h,\underline{u}] - 2a_h = 0 \ , \ h = 1,2,\ldots k. \qquad (3.4.24)$$

Our purpose is achieved, and our theorem has therefore been proved, if we take

$$a_j = \frac{1}{2} T_{11}[\underline{u}_j,\underline{u}] \ , \ j=1,2,\ldots k \ . \qquad (3.4.25)$$

Except for the first eigenvalue ω_1 and its associated eigenfunction $\underline{u}_1$, our ordered sequence of eigenvalues and associated eigenfunctions has been introduced through the variational equation (3.4.11). We have already seen that each eigenfunction $\underline{u}_j$ defines a stationary value ω_j of the quotient (3.4.12). We shall now prove that <u>we may also define the higher eigenvalues and their associated eigenfunctions as the solutions of a sequence of minimum problems similar to (3.4.2)</u>. The existence of solutions to the minimum problems in this sequence is of course tacitly assumed in this alternative definition. The only modification in our sequence of minimum problems, in comparison with our basic problem (3.4.2), consists of a restriction in the class of competing admissible displacement fields. In each minimum problem of our sequence we restrict the class of competing kinematically admissible displacement fields by the additional requirement that they must be orthogonal to the solutions of all preceding minimum problems. Our modified definition of ω_h as the solution of the h-th minimum problem reads therefore

$$\omega_h = \text{Min}\,\frac{P_2[\underline{u}]}{T_2[\underline{u}]} \qquad (3.4.26)$$

subject to the subsidiary conditions or <u>side conditions</u>

$$T_{11}[\underline{u}_j,\underline{u}] = 0\ ,\ j=1,2,\ldots(h-1)\ . \qquad (3.4.27)$$

We may normalise the solution $\underline{u}_h$ of the problem by requiring

$$T_2[\underline{u}_h] = 1\ . \qquad (3.4.28)$$

Before we prove that the minima ω_h and the solutions $\underline{u}_h$ coincide with the eigenvalues ω_h and associated eigenfunctions as defined previously, we observe that the sequence of minima (3.4.26) is necessarily non-decreasing

$$\omega_1 \leqq \omega_2 \leqq \omega_3 \leqq \cdots , \qquad (3.4.29)$$

due to the increasing number of restrictions imposed on the competing displacement fields. We also remember that the first minimum (3.4.26) is defined without side conditions and is therefore necessarily equal to the first eigenvalue, originally defined by (3.4.2). Our proof proceeds now by complete induction. If the coincidence of ω_j and $\underline{u}_j$ according to both definition is assumed up to $j=(h-1)$, we shall prove that it also holds for $j=h$. Since we are assured of the equivalence for $j=1$, we are in possession of the required starting point for a proof by induction.

Let $\underline{u}_h$ denote the normalised solution of (3.4.26) and (3.4.27), and let $\underline{\eta}$ denote an <u>arbitrary</u> kinematically admissible displacement field subject to the <u>same</u> conditions of orthogonality

(3.4.27), i.e.

$$T_{11}[\underline{u}_j,\underline{\eta}] = 0 \ , \ j=1,2,\ldots(h-1) \ . \qquad (3.4.30)$$

The minimum property of (3.4.26) is then expressed by an inequality, similar to (3.4.4),

$$\frac{P_2[\underline{u}_h+\varepsilon\underline{\eta}]}{T_2[\underline{u}_h+\varepsilon\underline{\eta}]} \geqq \frac{P_2[\underline{u}_h]}{T_2[\underline{u}_h]} = \omega_h \qquad (3.4.31)$$

for <u>arbitrary</u> values of the constant ε. We rewrite this inequality in a form similar to (3.4.9)

$$\varepsilon\{P_{11}[\underline{u}_h,\underline{\eta}] - \omega_h T_{11}[\underline{u}_h,\underline{\eta}]\} + \varepsilon^2\{P_2[\underline{\eta}] - \omega_h T_2[\underline{\eta}]\} \geqq 0 \ . \qquad (3.4.32)$$

Since the second term is non-negative for all $\underline{\eta}$ which satisfy (3.4.30), a necessary and sufficient condition resulting from this inequality is

$$P_{11}[\underline{u}_h,\underline{\eta}] - \omega_h T_{11}[\underline{u}_h,\underline{\eta}] = 0 \qquad (3.4.33)$$

for <u>every</u> kinematically admissible displacement field $\underline{\eta}$ which satisfies the orthogonality relations (3.4.30).

Our equation (3.4.33) bears a close resemblance to the variational equation (3.4.11), but it is clearly distinct from the latter equation in view of the restrictions imposed on $\underline{\eta}$ in (3.4.33). We shall prove, however, that (3.4.33) remains valid, if $\underline{\eta}$ is replaced by any kinematically admissible displacement field $\underline{\zeta}$ <u>not</u> subject to restrictions (3.4.30). It will be remembered from (3.4.23) that we may always write

$$\underline{\zeta} = \sum_{j=1}^{h-1} t_j \underline{u}_j + \underline{\eta} \ , \tag{3.4.34}$$

where $\underline{\eta}$ satisfies (3.4.30). To this end we have to choose the coefficients t_j corresponding to (3.4.25)

$$t_j = \frac{1}{2} T_{11}[\underline{u}_j, \underline{\zeta}] \ , \ j=1,2,\ldots(h-1). \tag{3.4.35}$$

We now replace $\underline{\eta}$ in the left-hand member of (3.4.33) by $\underline{\zeta}$, substitute from (3.4.34) and evaluate the resulting expression. We find

$$P_{11}[\underline{u}_h, \underline{\zeta}] - \omega_h T_{11}[\underline{u}_h, \underline{\zeta}] =$$

$$= \sum_{j=1}^{(h-1)} t_j \{P_{11}[\underline{u}_h, \underline{u}_j] - \omega_h T_{11}[\underline{u}_h, \underline{u}_j]\} + P_{11}[\underline{u}_h, \underline{\eta}] - \omega_h T_{11}[\underline{u}_h, \underline{\eta}]. \tag{3.4.36}$$

The last pair of terms cancel each other on account of (3.4.33). The remaining terms vanish separately. This is obvious for the terms $T_{11}[\underline{u}_h, \underline{u}_j]$ from the orthogonality conditions (3.4.27) imposed on $\underline{u}_h$. We also know that the variational equation (3.4.11) is satisfied by $\omega=\omega_j$, $\underline{u}=\underline{u}_j$ $(j=1,2,\ldots(h-1)$. It follows from this equation, by taking $\underline{\zeta} = \underline{u}_h$, that $P_{11}[\underline{u}_h, \underline{u}_j]$ vanishes simultaneously with $T_{11}[\underline{u}_h, \underline{u}_j]$. Hence we have from (3.4.36) and (3.4.33)

$$P_{11}[\underline{u}_h, \underline{\zeta}] - \omega_h T_{11}[\underline{u}_h, \underline{\zeta}] = 0 \ , \tag{3.4.37}$$

and we have proved that ω_h and $\underline{u}_h$, defined by the minimum problem (3.4.26) and (3.4.27), are solutions of the variational equation

(3.4.11). This completes our proof of the equivalence of our two alternative definitions of ω_h and $\underline{u}_h$ as eigenvalues and eigenfunctions of (3.4.11) on the one hand, and as minima (3.4.26) and minimising displacement fields on the other hand.

Earlier in this section it was remarked that we may replace $T_2[\underline{u}]$ in the denominator of the minimum problem (3.4.2) by any positive definite quadratic functional $T_2^*[\underline{u}]$ whose typical integrand is not necessarily definite. An investigation of the modified minimum problem (3.4.3) yields the same information on the stability of the fundamental state as an investigation of the original problem (3.4.2). The analysis of the latter problem, starting with (3.4.4) and ending with equation (3.4.37) may be repeated word for word in the case of the minimum problem (3.4.3), if we replace the quadratic and bilinear functionals $T_2[\underline{u}]$ and $T_{11}[\underline{v},\underline{w}]$, the eigenfunctions $\underline{u}_h$ and the eigenvalues ω_h by their corresponding starred counterparts. If we take in particular the case where $T_2^*[\underline{u}]$ is the denominator in Rayleigh's principle, we recover the pertinent well-known theorems for small free vibrations about the configuration of equilibrium in state I[e.g.5].

The reader will have noticed that our discussion in the present section has been based entirely on functional relations, and in particular on the variational equation (3.4.11). For the basic notions involved we have drawn heavily on the fundamental treatise by Courant and Hilbert [5,6]. We have two reasons for our preference of the functional approach in the present general discussion over the more familiar eigenvalue theory of

homogeneous boundary problems for linear differential equations. Our most fundamental reason is that the energy criterion for the stability of equilibrium of continuous bodies represents essentially a concept of functional analysis. Even though the discussion of the variational equation (3.4.11) might admittedly be replaced by a similar discussion of the equivalent differential equations and boundary conditions, the continuation of the investigation in the critical case $\omega_1=0$ in sections 3.6 to 3.8 would still require a return to the functional formulation. A second reason for our preference is the far greater simplicity of the discussion in terms of functional analysis, in comparison with a discussion of the equivalent differential equations and boundary conditions. All relevant conditions are embraced by a single variational equation. Once the few required basic concepts of functional analysis have been grasped, the analysis is conspicuously brief.

Even if our purpose of a general discussion is served best by a formulation in terms of concepts of functional analysis, a better understanding of its implications may well be furthered by a transcription into terms of the more familiar concepts of differential equations and boundary conditions. We prefer to postpone such a transcription until the end of the next section, where we shall derive from our variational equation (3.4.11) the differential equations and boundary conditions for neutral equilibrium for the two examples discussed previously.

3.5 The stability limit, neutral equilibrium and buckling modes.

In the last section we have seen that the conditions for the stability of equilibrium in the fundamental state I may be formulated in terms of the solution of the minimum problem (3.4.2). A necessary condition for stability is $\omega_1 \geqq 0$, a sufficient condition is $\omega_1 > 0$.

No decision has as yet been obtained in the limiting case $\omega_1 = 0$. This case will be called a critical case of neutral equilibrium, and the equilibrium is said to be at the stability limit in this case. The stability limit is characterized by

$$\omega_1 = \text{Min.} \frac{P_2[\underline{u}]}{T_2[\underline{u}]} = 0 \ . \qquad (3.5.1)$$

The normalised minimising displacement field or associated eigenfunction $\underline{u}_1$ will be called the critical buckling mode. We may also describe the case of equilibrium at the stability limit in terms of the semi-definite second variation alone

$$P_2[\underline{u}] \geqq 0 \ , \ P_2[\underline{u}_1] = 0 \ , \qquad (3.5.2)$$

where $\underline{u}_1$ is the critical buckling mode. Finally, our general variational equation (3.4.11) takes the simplified form that the buckling mode $\underline{u}_1$ is a solution of the variational equation

$$P_{11}[\underline{u},\underline{\zeta}] = 0 \ , \qquad (3.5.3)$$

holding for every kinematically admissible displacement field $\underline{\zeta}$.

In the continued investigation of stability in a critical case of neutral equilibrium, to be discussed in sections 3.6 to 3.8, it is essential to know first all possible critical

buckling modes, i.e. all displacement fields $\underline{u}$ for which the second variation $P_2[\underline{u}]$ vanishes. In order to answer the question whether the second variation may also be zero for a displacement field $\underline{u}$ which is not a multiple of $\underline{u}_1$, we write (cf. (3.4.23))

$$\underline{u} = a\underline{u}_1+\underline{v} \ , \text{ where } T_{11}[\underline{u}_1,\underline{v}] = 0 \ . \tag{3.5.4}$$

We have now from (3.5.2) and (3.5.3)

$$P_2[\underline{u}] = P_2[a\underline{u}_1+\underline{v}] = a^2P_2[\underline{u}_1] + ap_{11}[\underline{u}_1,\underline{v}] + P_2[\underline{v}] = P_2[\underline{v}] \ . \tag{3.5.5}$$

Hence, if $P_2[\underline{u}]$ vanishes for a displacement field $\underline{u}$ which is not a multiple of the critical buckling mode $\underline{u}_1$, it must also vanish for the displacement field $\underline{v}$ orthogonal to this buckling mode, defined by (3.5.4).

We now recall our sequence of minimum problems (3.4.26), (3.4.27), in particular

$$\omega_2 = \text{Min.} \frac{P_2[\underline{u}]}{T_2[\underline{u}]} \tag{3.5.6}$$

under the side condition

$$T_{11}[\underline{u}_1,\underline{u}] = 0 \ . \tag{3.5.7}$$

If ω_2 is positive, no displacement field $\underline{v}$ orthogonal to $\underline{u}_1$ exists for which the second variation is zero. In this case the critical buckling mode $\underline{u}_1$ is called a simple critical buckling mode. On the other hand, if $\omega_2=\omega_1=0$, a second critical buckling mode exists in the form of the eigenfunction $\underline{u}_2$ associated with the second eigenvalue $\omega_2=0$.

In the latter case of a so-called multiple critical buckling mode, we must of course continue our investigation. We now have to answer the question whether displacement fields $\underline{u}$ exist which are linearly independent of both critical buckling modes and for which the second variation again vanishes. By a similar argument as before we may prove that the answer is provided by the solution ω_3 of the third minimum problem in the sequence (3.4.26), (3.4.27). If ω_3 is positive, no displacement field $\underline{u}$ exists which is linearly independent of $\underline{u}_1$ and $\underline{u}_2$, and for which $P_2[\underline{u}]$ is zero. Evidently we have to continue our investigation again, if $\omega_3=0$.

Our results may be summarized by a general prescription of the procedure to be followed in the analysis. The sequence of minimum problems (3.4.26), (3.4.27) has to be investigated until a positive minimum, say $\omega_{(h+1)}$, is obtained, or at least has been shown to exist. The most general displacement field $\underline{u}$ for which the second variation $P_2[\underline{u}]$ vanishes is then given by a linear combination of h critical buckling modes

$$\underline{u} = \sum_{j=1}^{h} a_j \underline{u}_j \ . \qquad (3.5.8)$$

It will be convenient to refer to a coefficient a_j in (3.5.8) as the amplitude of the buckling mode $\underline{u}_j$.

It will be noticed that the vanishing of the second variation $P_2[\underline{u}]$ for a displacement field (3.5.8) in the case of a multiple eigenvalue $\omega_1=\omega_2= \ldots =\omega_h=0$, is an immediate consequence of the discussion of the variational equation (3.4.11) in the

previous section. It had not been proved previously, however, that the second variation cannot vanish for other displacement fields. The additional analysis in the present section was indeed required to establish this important feature.

The conventional approach to the theory of elastic stability starts from the concept of neutral equilibrium [e.g. 1,2,15]. In this approach, a fundamental state of equilibrium is called neutral, if additional configurations of equilibrium exist which are obtained from the fundamental state by means of infinitesimal additional displacements. The differential equations and boundary conditions which express the existence of such adjacent configurations of equilibrium, are linear and homogeneous in the additional displacements from the fundamental state. These equations are the complete equivalent of our variational equation (3.5.3)*). This statement is easily verified by means of (3.2.3) and (3.2.6). Neglecting higher order terms in the potential energy in an adjacent state II, obtained by the infinitesimal displacement field $\underline{u}$ from the fundamental state I, we have for the energy in state II

$$P_{II} = P_I + P_2[\underline{u}] \tag{3.5.9}$$

The condition of equilibrium in this state is given by the condition for a stationary value of the energy, i.e. the variational equation (3.5.3) holding for any kinematically admissible displacement field ζ.

*) In spite of occasional claims to the contrary [e.g. 13], the approach through the concept of neutral equilibrium is subject to the same restrictions as the energy criterion: a justification of the neutral equilibrium approach to stability theory is possible only for conservative systems (cf. sections 1.4 and 2. ...).

The equation of neutral equilibrium (3.5.3) is satisfied if one of the eigenvalues of our variational equation (3.4.11) is zero. It is immaterial from the viewpoint of neutral equilibrium according to its classical definition, whether or not the eigenvalue zero is the smallest eigenvalue of (3.4.11). In the first-mentioned case we have $\omega_1=0$, and we are dealing with a critical case of neutral equilibrium as defined earlier in this section. In all other cases of neutral equilibrium we have $\omega_1 < 0$, and we know already from our energy considerations on the basis of the second variation alone that equilibrium is actually unstable. It is somewhat unfortunate that the conventional notion of neutral equilibrium does not distinguish clearly between the critical case, where the second variation of the energy alone is incapable of supplying a decision on stability, and the other cases, where the second variation has already established instability. The critical case of neutral equilibrium is by far the most important one, and whenever there is no danger of confusion with other cases of neutral equilibrium, we shall omit the words "critical case" in referring to the neutral equilibrium under consideration.

We now return one again to the two examples which have been discussed before as illustrations of the general theory. We shall derive for these examples the differential equations and boundary conditions which are the equivalent of the variational equation of neutral equilibrium (3.5.3).

In the case of the bar under compressive end loads, the second variation of the energy is given by (3.2.11). The geometric boundary conditions are in this case $w(0)=w(\ell)=0$. The explicit form of variational equation (3.5.3) is

$$P_{11}[w,\zeta] = \int_0^{\ell} [Bw''\zeta''-Nw'\zeta']dx = 0 \ , \qquad (3.5.10)$$

holding for every twice continuously differentiable function $\zeta(x)$ which vanishes for $x=0$ and $x=\ell$. We proceed to remove the derivatives of the arbitrary function $\zeta(x)$ under the sign of integration. This purpose may be accomplished through integration by parts, if it is assumed that the solution of (3.5.10) has continuous derivatives up to and including the fourth order. The result of a repeated integration by parts is then

$$Bw''\zeta'\Big|_0^{\ell} - [(Bw'')' + Nw']\zeta\Big|_0^{\ell} + \int_0^{\ell} [(Bw'')'' + (Nw')']\zeta \, dx = 0 \ . \qquad (3.5.11)$$

The first two terms in (3.5.11), which depend only on the boundary values of the functions concerned and their derivatives will be called boundary terms. In view of the kinematic conditions imposed on $\zeta(x)$, that is $\zeta(0)=\zeta(\ell)=0$, the second boundary term vanishes identically. Since both N and B are constants, equation (3.5.11) reduces to

$$Bw''\zeta'\Big|_0^{\ell} + \int_0^{\ell}[Bw'''' + Nw'']\zeta \, dx = 0 \ . \qquad (3.5.12)$$

The remaining argument is based on a fundamental lemma of the calculus of variations [5,ch.4]. The importance of this

lemma for our subsequent analysis warrants a comprehensive statement and proof here. Let $\underline{x}$ denote a set of n independent variables $x_i (i=1,2,\ldots n)$, and let $f(\underline{x})$ denote a function of these variables, continuous in a closed domain D of n-dimensional $\underline{x}$-space. Let C_k $(k=0,1,2,..)$ denote the class of functions $\zeta(\underline{x})$ in the closed domain D which have continuous partial derivatives up to and including the order k. This class may possibly also be restricted by certain homogeneous boundary conditions*) on the boundary of D. If the relation

$$\int_D f(\underline{x})\zeta(\underline{x})d\underline{x} = 0 \tag{3.5.13}$$

holds for <u>every</u> function $\zeta(\underline{x})$ of class C_k, it follows that

$$f(\underline{x}) = 0 \tag{3.5.14}$$

<u>everywhere in D</u>. An indirect proof of this basic lemma is easily given. Suppose it were false, $f(\underline{x})$ would be non-zero, say positive, at some point $\underline{y}$ in D. The continuity of $f(\underline{x})$ implies that this function is then also positive in some n-dimensional sphere of radius r about the center $\underline{y}$. We may choose a function $\zeta(\underline{x})$ of class C_k, defined by

$$\left.\begin{aligned} &\zeta(\underline{x}) = [r^2 - \sum_{i=1}^{n}(x_i-y_i)^2]^{k+1} \text{ for } \sum_{i=1}^{n}(x_i-y_i)^2 \leqq r^2, \\ &\zeta(\underline{x}) = 0 \quad \text{everywhere else in D.} \end{aligned}\right\} \tag{3.5.15}$$

*) The theorem remains valid, if non-homogeneous boundary conditions are imposed on $\zeta(\underline{x})$, but a complete proof is more complicated in this case.

This function $\zeta(\underline{x})$ is positive inside the sphere of radius r about $\underline{y}$, and zero everywhere else. The integrand of (3.5.13) is therefore positive inside this sphere, and zero everywhere else in D. The integral in the left-hand member of (3.5.13) is now also positive, and we have arrived at a contradiction. It follows that our assumption of a non-zero value of $f(\underline{x})$ at the point $\underline{y}$ is untenable, and the lemma has thus been proven.

In order to apply the fundamental lemma to (3.5.12), we consider the class of twice continuously differentiable functions $\zeta(x)$ for which both the function and its first derivative vanish at $x=0$ and $x=\ell$. The boundary term in (3.5.12) then disappears, and we obtain the well-known homogeneous and linear <u>differential equation of neutral equilibrium</u> for a bar under compressive end loads

$$Bw'''' + Nw'' = 0 \ . \tag{3.5.16}$$

The integral in (3.5.12) now vanishes identically. Removing our artificial restriction $\zeta'(0) = \zeta'(\ell) = 0$, which is <u>not</u> imposed by any geometric condition of the problem, we infer that the cofactor of ζ' in the boundary term must also vanish. Hence we obtain the homogeneous and linear <u>dynamic or natural boundary conditions</u>

$$x=0 \text{ and } x=\ell: \quad Bw'' = 0 \ , \tag{3.5.17}$$

in addition to the homogeneous and linear <u>geometric boundary conditions</u>

$$x=0 \text{ and } x=\ell: \quad w = 0 \ , \tag{3.5.18}$$

imposed on all competing functions $w(x)$.

The solution of the equation and boundary conditions for neutral equilibrium is extremely simple in the present example because (3.5.16) is a homogeneous linear differential equation with constant coefficients. Non-trivial solutions which satisfy the boundary conditions exist only for special values of the compressive load, the so-called <u>buckling loads</u> [17, §2.2]

$$N = N_k \frac{k^2\pi^2 B}{\ell^2}, \quad k=1,2,\ldots \tag{3.5.19}$$

The associated solutions are called <u>buckling modes</u>. They are given by

$$w = W_k \sin k\pi x/\ell , \tag{3.5.20}$$

where W_k is an arbitrary constant. In order to decide whether the neutral equilibrium at a buckling load (3.5.19) is a <u>critical case of neutral equilibrium</u>, we have to investigate the second variation (3.2.11) itself. This is again quite simple in the present example. Since the competing deflections $w(x)$ in (3.2.11) are at least twice continuously differentiable, the Fourier sine series of an arbitrary deflection function satisfying (3.5.17) and (3.5.18)

$$w(x) = \sum_{k=1}^{\infty} W_k \sin k\pi x/\ell \tag{3.5.21}$$

may be differentiated twice term by term. The second variation (3.2.11) may then be written in the form

$$P_2[w] = \frac{\pi^2}{4\ell} \sum_{k=1}^{\infty} k^2 [\frac{k^2\pi^2 B}{\ell^2} - N]\, W_k^2 . \tag{3.5.22}$$

It is positive definite for $N < N_1$, semi-definite for $N=N_1$ and indefinite for $N > N_1$. Hence $N=N_1$ defines a critical case of neutral equilibrium, and this load is the critical buckling load or Euler buckling load. At the higher buckling loads $N_k (k \geq 2)$ we have unstable cases of neutral equilibrium.

We now turn to our second example, the three-dimensional elastic body under dead loads. The second variation of the potential energy in this case is defined by (3.2.17), and the geometric boundary conditions for a kinematically admissible displacement field are $\underline{u}=0$ on S_u. It is convenient to introduce the concept of a (symmetric) fictitious linear strain tensor

$$\theta_{ij} = \frac{1}{2}(u_{i,j}+u_{j,i}) \ . \tag{3.5.23}$$

This fictitious strain tensor should be clearly distinguished from the actual strain tensor γ_{ij}, defined by (3.1.7), which describes the actual deformation of the body in the transition from state I to state II. The fictitious strain tensor coincides with the actual strain tensor, if and only if the displacement field $\underline{u}$ from the fundamental state I to state II is infinitesimal. Likewise, it is convenient to denote the second term in the integrand of (3.2.17)

$$\overline{A} = G[\theta_{ij}\theta_{ij} + \frac{\nu}{1-2\nu}(\theta_{hh})^2] \tag{3.5.24}$$

as the fictitious elastic strain energy density, associated with the fictitious linear strain tensor θ_{ij} through Hooke's law. It coincides with the actual elastic strain energy density, if and

only if the fundamental state I is the undeformed state and the displacement field $\underline{u}$ from this fundamental state to state II is infinitesimal. Finally, we introduce the (symmetric) <u>fictitious stress tensor</u>, associated with the fictitious linear strain tensor through Hooke's law,

$$\sigma_{ij} = \frac{\partial \bar{A}}{\partial \Theta_{ij}} = 2G[\Theta_{ij} + \frac{\nu}{1-2\nu}\,\Theta_{hh}g_{ij}] , \qquad (3.5.25)$$

where g_{ij} is the unit tensor ($g_{ij}=0$ for $i\neq j$, $g_{ij}=1$ for $i=j$). Here again the fictitious stress tensor coincides with the actual stress tensor, if and only if the fundamental state I is the undeformed state and the displacement field $\underline{u}$ from the fundamental state to state II is infinitesimal.

After these preliminaries we may write down the explicit form of our variational equation (3.5.3) for neutral equilibrium

$$P_{11}[\underline{u},\underline{\zeta}] = \int_V [S_{ij}u_{h,i}\zeta_{h,j} + \sigma_{ij}\zeta_{i,j}]dv = 0 , \qquad (3.5.26)$$

where appropriate use has been made of the symmetry of both the initial stress tensor S_{ij} and the fictitious stress tensor σ_{ij}. Equation (3.5.26) must hold for every once continuously differentiable kinematically admissible displacement field $\underline{\zeta}$. We may again remove the derivatives of the components ζ_i from our integral by applying Green's theorem. To this end we have to assume that the displacement fields $\underline{u}$ which satisfy (3.5.26) are twice continuously differentiable. On this assumption we obtain

$$\int_S [S_{ij}u_{h,i} + \sigma_{hj}]n_j\zeta_h dS + \int_V [S_{ij}u_{h,i} + \sigma_{hj}]_{,j}\,\zeta_h dv = 0\ , \qquad (3.5.27)$$

where $\underline{n}$ is the unit outward normal vector to S. Restricting our attention first to displacement fields $\underline{\zeta}$ which vanish on S, we obtain from the fundamental lemma of the calculus of variations the homogeneous and linear <u>differential equations of neutral equilibrium</u> [19]

$$[S_{ij}u_{h,i} + \sigma_{hj}]_{,j} = 0\ . \qquad (3.5.28)$$

Returning to (3.5.27) and remembering that no geometric boundary conditions are imposed on $\underline{\zeta}$ on the part S_p of the surface, we obtain the homogeneous and linear <u>dynamic or natural boundary conditions</u>

$$[S_{ij}u_{h,i} + \sigma_{nj}]n_j = 0 \quad \text{on} \quad S_p\ . \qquad (3.5.29)$$

In addition to these we have the <u>geometric boundary conditions</u>

$$u_h = 0 \quad \text{on} \quad S_u\ . \qquad (3.5.30)$$

A slightly different form of the differential equations (3.5.28) may be obtained by appropriate use of the original equations of equilibrium (3.2.20) for the initial stresses in state I. The modified equations read

$$\sigma_{hj,j} + S_{ij}u_{h,ij} - X_i u_{h,i} = 0\ . \qquad (3.5.31)$$

The boundary conditions (3.5.29) may be reduced in a similar manner, by means of the original boundary conditions (3.2.21) for S_{ij} on S_p, to

$$\sigma_{hj}n_j + p_i u_{h,i} = 0 \quad \text{on} \quad S_p\ . \qquad (3.5.32)$$

The derivation of the equations and boundary conditions for neutral equilibrium suffices for our present purpose to demonstrate the appropriate manipulation of the general variational equation (3.5.3) in the example of an elastic body. The reader is referred to chapter 6 for a more detailed discussion of neutral equilibrium in this case.

3.6 Stability in a critical case of neutral equilibrium.

For the further investigation of stability in a critical case of neutral equilibrium it will be convenient to consider first the case of a simple critical buckling mode $\underline{u}_1$. The solution ω_2 of the minimum problem (3.5.6), (3.5.7) is then positive. Writing an arbitrary kinematically admissible displacement field $\underline{u}$ in the form (3.4.23)

$$\underline{u} = a\underline{u}_1 + \underline{v} \text{ , where } T_{11}[\underline{u}_1, \underline{v}] = 0 \text{ ,} \tag{3.6.1}$$

we are thus ensured of the inequality

$$P_2[\underline{v}] \geqq \omega_2 T_2[\underline{v}] \text{ ,} \tag{3.6.2}$$

where ω_2 is a positive constant.

Our reason for writing an arbitrary kinematically admissible displacement field in the form (3.6.1) is our conjecture that only "small" deviations $\underline{v}$ from a multiple $a\underline{u}_1$ of the buckling mode need to be considered. In fact, it is not difficult to prove by an argument similar to our discussion in section 3.3 that (3.6.2) ensures a positive energy increment (3.1.1) for sufficiently small values of the amplitude a in (3.6.1), provided that $a^{-2}T_2[\underline{v}]$ has a non-vanishing limit for $a \rightarrow 0$. Hence a further investigation is only required for those deviations $\underline{v}$ from a multiple $a\underline{u}_1$ of the buckling mode for which $a^{-2}T_2[\underline{v}]$ approaches zero if a tends to zero. A detailed proof of these statements may be omitted here because the following analysis does not presuppose any restriction on the order of magnitude of $\underline{v}$ in comparison with $a\underline{u}_1$.

In section 3.1 the existence of some positive constants $g, g', g'', \ldots$ with the property that the inequalities (3.1.2) ensure a non-negative energy increase (3.1.1) for every kinematically admissible displacement field $\underline{u}$, was established as a necessary and sufficient condition for the stability of equilibrium in the fundamental state I. A slightly modified version of this criterion is appropriate for displacement fields written in the form (3.6.1). A necessary and sufficient condition for stability is the existence of some positive constants $A, \bar{g}, \bar{g}', \bar{g}'', \ldots$ with the property that the inequalities

$$|a| \leqq A \ , \ |v_i| \leqq \bar{g} \ , \ |v_{i,j}| \leqq \bar{g}' \ , \ |v_{i,jk}| \leqq \bar{g}'' \ , \ \ldots \qquad (3.6.3)$$

ensure a non-negative energy increase (3.1.1)

$$P[a\underline{u}_1 + \underline{v}] \geqq 0 \ . \qquad (3.6.4)$$

The equivalence of this alternative form with the original criterion is easily proved. The inequalities (3.6.3) evidently imply the inequalities (3.1.2), and it only remains to show that the converse is also true. From (3.4.25) we have

$$a = \frac{1}{2} T_{11}[\underline{u}_1, \underline{u}] \ , \qquad (3.6.5)$$

and it follows that a is finite since $\underline{u}$ and its derivatives are bounded. Returning to (3.6.1), we see that $\underline{v}$ and its derivatives are then also bounded simultaneously with $\underline{u}$ and its derivatives. Hence the inequalities (3.6.3) are indeed equivalent to the inequalities (3.1.2).

The arguments y^λ of the typical integrand $F(\underline{y})$ of $P[\underline{u}]$, defined in section 3.2, may be written in a form similar to (3.6.1)

$$y^\lambda = ay_1^\lambda + z^\lambda \ , \quad \text{or} \quad \underline{y} = a\underline{y}_1 + \underline{z} \ , \tag{3.6.6}$$

where y_1^λ represents these arguments corresponding to the critical buckling mode $\underline{u}_1$. We apply Taylor's formula in order to obtain an expansion of the typical integrand about the set of arguments $a\underline{y}_1$. Terminating this expansion at the third order remainder, we have

$$\begin{aligned} F(a\underline{y}_1+\underline{z}) = F(a\underline{y}_1) + z^\lambda F_{,\lambda}(a\underline{y}_1) + \frac{1}{2} z^\lambda z^\mu F_{,\lambda\mu}(a\underline{y}_1) + \\ + \frac{1}{6} z^\lambda z^\mu z^\nu F_{,\lambda\mu\nu}(a\underline{y}_1+\overline{\underline{z}}) \ , \end{aligned} \tag{3.6.7}$$

where the arguments in the third order derivative are given by

$$ay_1^\lambda + \overline{z}^\lambda = ay_1^\lambda + \theta^\lambda z^\lambda, \ (\lambda \text{ not summed}; \ 0 \leqq \theta^\lambda \leqq 1) \ . \tag{3.6.8}$$

We identify the first term in (3.6.7) as the typical integrand of $P[a\underline{u}_1]$. Collecting the integrals of the first terms in the integrands (3.6.7), and remembering (3.2.3), (3.2.6) and (3.5.2), we may express this fact by the symbolism

$$F(a\underline{y}_1) \Longrightarrow a^3P_3[\underline{u}_1] + a^4P_4[\underline{u}_1] + O(a^5) \ . \tag{3.6.9}$$

Next we expand the first and second derivatives, which appear in (3.6.7) at the set of arguments ay_1^λ, by means of Taylor's formula

$$F_{,\lambda}(a\underline{y}_1) = F_{,\lambda}(0) + ay_1^{\mu} F_{,\lambda\mu}(0) + \frac{1}{2} a^2 y_1^{\mu} y_1^{\nu} F_{,\lambda\mu\nu}(0) +$$

$$+ \frac{1}{6} a^3 y_1^{\mu} y_1^{\nu} y_1^{\rho} F_{,\lambda\mu\nu\rho}(0) + \frac{1}{24} a^4 y_1^{\mu} y_1^{\nu} y_1^{\rho} y_1^{\sigma} F_{,\lambda\mu\nu\rho\sigma}(0) + O(a^5) , \tag{3.6.10}$$

$$F_{,\lambda\mu}(a\underline{y}_1) = F_{,\lambda\mu}(0) + ay_1^{\nu} F_{,\lambda\mu\nu}(\theta a\underline{y}_1) , \quad (0 \leqq \theta \leqq 1) . \tag{3.6.11}$$

We collect the integrals whose integrands are linear in $\underline{z}$ and homogeneous of degrees zero to four in the amplitude a of the buckling mode. Remembering (3.2.6), (3.4.6) and (3.5.1), we note the symbolic relations

$$z^{\lambda} F_{,\lambda}(0) \Longrightarrow P_1[\underline{v}] = 0 , \tag{3.6.12}$$

$$az^{\lambda} y_1^{\mu} F_{,\lambda\mu}(0) \Longrightarrow aP_{11}[\underline{u}_1, \underline{v}] = 0 , \tag{3.6.13}$$

$$\frac{1}{2} a^2 z^{\lambda} y_1^{\mu} y_1^{\nu} F_{,\lambda\mu\nu}(0) \Longrightarrow a^2 P_{21}[\underline{u}_1, \underline{v}] , \tag{3.6.14}$$

$$\frac{1}{6} a^3 z^{\lambda} y_1^{\mu} y_1^{\nu} y_1^{\rho} F_{,\lambda\mu\nu\rho}(0) \Longrightarrow a^3 P_{31}[\underline{u}_1, \underline{v}] , \tag{3.6.15}$$

$$\frac{1}{24} a^4 z^{\lambda} y_1^{\mu} y_1^{\nu} y_1^{\rho} y_1^{\sigma} F_{,\lambda\mu\nu\rho\sigma}(0) \Longrightarrow a^4 P_{41}[\underline{u}_1, \underline{v}] . \tag{3.6.16}$$

Finally, we identify the term which is homogeneous and quadratic in z, and independent of a as the typical integrand of $P_2[\underline{v}]$, i.e. the symbolic relation

$$\frac{1}{2} z^{\lambda} z^{\mu} F_{,\lambda\mu}(0) \Longrightarrow P_2[\underline{v}] . \tag{3.6.17}$$

All terms in (3.6.7) have now been identified, except the last, cubic term in z and that part of the second, quadratic term in z whose coefficient is given by the last term in (3.6.11). We write these terms together in the form $\overline{A}_{\lambda\mu} z^{\lambda} z^{\mu}$, where the coefficients $\overline{A}_{\lambda\mu}$ are specified by

$$\bar{A}_{\lambda\mu} = \frac{1}{6}\, z^{\nu} F_{,\lambda\mu\nu}(a\underline{y}_1+\underline{\bar{z}}) + \frac{1}{2}\, a y_1^{\nu}\, F_{,\lambda\mu\nu}(\Theta a\underline{y}_1) \quad . \tag{3.6.18}$$

Denoting the sum of integrals of these remaining terms by $R_2[\underline{v}]$, we have the symbolic relation

$$\bar{A}_{\lambda\mu} z^{\lambda} z^{\mu} \Longrightarrow R_2[\underline{v}] \quad . \tag{3.6.19}$$

By means of (3.6.9), (3.6.12) - (3.6.17) and (3.6.19) we may now write for the energy increase (3.1.1) in the kinematically admissible displacement field (3.6.1)

$$P[a\underline{u}_1+\underline{v}] = a^3 P_3[\underline{u}_1] + a^4 P_4[\underline{u}_1] + a^2 P_{21}[\underline{u}_1,\underline{v}] + a^3 P_{31}[\underline{u}_1,\underline{v}] +$$

$$+ a^4 P_{41}[\underline{u}_1,\underline{v}] + P_2[\underline{v}] + R_2[\underline{v}] + O(a^5) \quad . \tag{3.6.20}$$

If we restrict our displacement field to a multiple of the buckling mode, i.e. if we take $\underline{v} \equiv 0$, we obtain

$$P[a\underline{u}_1] = a^3 P_3[\underline{u}_1] + a^4 P_4[\underline{u}_1] + O(a^5) \quad . \tag{3.6.21}$$

In view of the condition of a non-negative energy increase in a stable fundamental state, we recover the <u>necessary</u> conditions of stability (3.2.8) and (3.2.9). A continued investigation is, of course, only called for if these necessary conditions are satisfied. Henceforward we may therefore assume in particular

$$P_3[\underline{u}_1] = 0 \quad , \tag{3.6.22}$$

and we may omit the first term in (3.6.20).

Our next step is to show that the <u>quadratic remainder $R_2[\underline{v}]$ in (3.6.20) may be made as small as we please in comparison with $P_2[\underline{v}]$, by choosing the positive constants in (3.6.3) sufficiently small</u>. This is indeed the primary purpose of our

reduction of the energy expression to the form (3.6.20). The actual proof of our assertion is again comparatively simple. It follows exactly along the same lines as our proof in section 3.3 that a positive greatest lower bound for the quotient of $P_2[\underline{u}]$ and $T_2[\underline{u}]$ is a sufficient condition for stability. The homogeneous quadratic form $\overline{A}_{\lambda\mu}z^\lambda z^\mu$ satisfies inequality (3.3.13), where $A_{\lambda\mu}$ has been replaced by $\overline{A}_{\lambda\mu}$ and y^λ by z^λ. The coefficients $\overline{A}_{\lambda\mu}$, given by (3.6.18) may be made as small in modulus as we please by choosing our positive constants in (3.6.3) sufficiently small. By a suitable choice of these constants, the absolute value of $\overline{A}_{\lambda\mu}z^\lambda z^\mu$ may be reduced to an arbitrarily small positive fraction of $C_{\lambda\mu}z^\lambda z^\mu$, the typical integrand of $T_2[\underline{v}]$, and we obtain for the integrals in question the inequality

$$|R_2[\underline{v}]| \leqq \varepsilon' T_2[\underline{v}] , \tag{3.6.23}$$

where ε' is a positive constant as small as we please. By means of (3.6.2) we obtain the desired result

$$|R_2[\underline{v}]| \leqq \varepsilon P_2[\underline{v}] , \tag{3.6.24}$$

where $\varepsilon = \varepsilon'/\omega_2$ is another positive constant <u>as small as we please</u>. Substituting the estimate (3.6.24) into (3.6.20), we have the inequalities

$$P[a\underline{u}_1+\underline{v}] \gtreqless a^4P_4[\underline{u}_1] + a^2P_{21}[\underline{u}_1,\underline{v}] + a^3P_{31}[\underline{u}_1,\underline{v}] + a^4P_{41}[\underline{u}_1,\underline{v}] + \\ + (1\mp\varepsilon)P_2[\underline{v}] + O(a^5) , \tag{3.6.25}$$

where either <u>both upper signs</u> (> and -) or <u>both lower signs</u> (< and +) have to be combined.

We shall now investigate the functional of the displacement field $\underline{v}$ orthogonal to $\underline{u}_1$, defined by

$$P_2[\underline{v}] + a^2 P_{21}[\underline{u}_1,\underline{v}] + a^3 P_{31}[\underline{u}_1,\underline{v}] + a^4 P_4[\underline{u}_1,\underline{v}] \ , \quad (3.6.26)$$

for a prescribed arbitrary constant value of the amplitude a of the buckling mode. In particular, we shall determine the minimum*) of (3.6.26) for any given value of a. Although the functionals of $\underline{v}$ which appear in (3.6.25), viz.

$$(1\mp\varepsilon)P_2[\underline{v}] + a^2 P_{21}[\underline{u}_1,\underline{v}] + a^3 P_{31}[\underline{u}_1,\underline{v}] + a^4 P_{41}[\underline{u}_1,\underline{v}] \ , \quad (3.6.27)$$

are slightly different from (3.6.26), we may give a complete description of the behaviour of the functionals (3.6.27) in terms of the behaviour of (3.6.26). The sole difference between (3.6.26) and (3.6.27) is that the latter functionals have an additional factor $(1\mp\varepsilon)$ in the first, quadratic term; the linear terms are identical. If V denotes the value of (3.6.26) for some displacement field $\underline{v}$, then the displacement fields $\underline{v}/(1\mp\varepsilon)$ yield values $V/(1\mp\varepsilon)$ for the functionals (3.5.27), where again either the upper sign or the lower sign has to be applied everywhere. Without any loss in generality we may assume $\varepsilon < 1$, since ε is an arbitrarily small positive number. The minima of the functionals (3.6.27) are then also given by the minimum of (3.6.26) divided by $(1\mp\varepsilon)$.

*) Here again we shall not prove the existence of such a minimum, although it will be shown that a stationary value of (3.6.26) is necessarily a minimum.

Let $\underline{v}'$ denote the displacement field orthogonal to the buckling mode $\underline{u}_1$ for which (3.6.26) attains its minimum. Let $\underline{\eta}$ denote an arbitrary kinematically admissible displacement field subject to the same condition of orthogonality with respect to the buckling mode

$$T_{11}[\underline{u}_1,\underline{\eta}] = 0 \ . \tag{3.6.28}$$

The minimum property of (3.6.26) for the displacement field $\underline{v}'$ is then expressed by the inequality

$$\varepsilon''\{P_{11}[\underline{v}',\underline{\eta}] + a^2P_{21}[\underline{u}_1,\underline{\eta}] + a^3P_{31}[\underline{u}_1,\underline{\eta}] + a^4P_{41}[\underline{u}_1,\underline{\eta}]\} + \\ + \varepsilon''^2P_2[\underline{\eta}] \geq 0 \ , \tag{3.6.29}$$

holding for arbitrary values of the constant ε''. Since the second term is non-negative for all $\underline{\eta}$ which satisfy (3.6.28), the necessary and sufficient condition resulting from this inequality is that $\underline{v}=\underline{v}'$ satisfies the equation

$$P_{11}[\underline{v},\underline{\eta}] + a^2P_{21}[\underline{u}_1,\underline{\eta}] + a^3P_{31}[\underline{u}_1,\underline{\eta}] + a^4P_{41}[\underline{u}_1,\underline{\eta}] = 0 \tag{3.6.30}$$

for every kinematically admissible displacement field $\underline{\eta}$ which satisfies the orthogonality relation (3.6.28).

The existence of a solution of the linear variational equation (3.6.30) will be assumed. It follows from (3.6.29) that this solution defines an absolute minimum of the functional (3.6.26) and is therefore necessarily unique. Uniqueness of the solution of (3.6.30) may of course be confirmed indirectly. Thus assume that this equation has two distinct solutions, say $\underline{v}'$ and

$\underline{v}''$. Their difference would satisfy the equation

$$P_{11}[(\underline{v}'-\underline{v}''),\underline{\eta}] = 0 \ . \tag{3.6.31}$$

Taking $\underline{\eta} = (\underline{v}'-\underline{v}'')$ in (3.6.31), we obtain by means of (3.4.7)

$$2P_2[(\underline{v}'-\underline{v}'')] = 0 \ , \tag{3.6.32}$$

in contradiction to (3.6.2) unless $(\underline{v}'-\underline{v}'') \equiv 0$.

The actual evaluation of the minimum value of (3.6.26), once the variational equation (3.6.30) has been solved, may be simplified by appropriate use of this equation. If we take $\underline{\eta}$ equal to the minimizing displacement field $\underline{v}'$, we obtain by means of (3.4.7)

$$2P_2[\underline{v}'] + a^2P_{21}[\underline{u}_1,\underline{v}'] + a^3P_{31}[\underline{u}_1,\underline{v}'] + a^4P_{41}[\underline{u}_1,\underline{v}'] = 0. \tag{3.6.33}$$

The minimum value of (3.6.26) may therefore be evaluated by either of the following formulae, whichever may be most convenient in a particular case

$$\begin{aligned} \text{Min.}\{P_2[\underline{v}] + a^2P_{21}[\underline{u}_1,\underline{v}] + a^3P_{31}[\underline{u}_1,\underline{v}] + a^4P_{41}[\underline{u}_1,\underline{v}]\} &= \\ = \tfrac{1}{2}\{a^2P_{21}[\underline{u}_1,\underline{v}'] + a^3P_{31}[\underline{u}_1,\underline{v}'] + a^4P_{41}[\underline{u}_1,\underline{v}']\} &= \\ = -P_2[\underline{v}'] \ . \end{aligned} \tag{3.6.34}$$

The second formula is also of some general interest. It shows that the minimum of (3.6.26) is <u>always negative</u> (or at least non-positive, if the remote possibility that $\underline{v}'$ vanishes identically is also allowed for).

The <u>linearity</u> of equation (3.6.30) implies that we may always write its solution in the form

$$\underline{v}' = a^2\underline{v}_2 + a^3\underline{v}_3 + a^4\underline{v}_4 \tag{3.6.35}$$

where the (vector) function $\underline{v}_2$ is the unique solution of the equation

$$P_{11}[\underline{v},\underline{\eta}] + P_{21}[\underline{u}_1,\underline{\eta}] = 0 \tag{3.6.36}$$

which satisfies the orthogonality condition $T_{11}[\underline{u}_1,\underline{v}] = 0$. We omit the similar equations for $\underline{v}_3$ and $\underline{v}_4$ because we shall not need their explicit form. Substituting (3.6.35) into (3.6.34) we may write the result of our analysis of the minimum of the functional (3.6.26) in the form

$$\begin{aligned} \text{Min.}\{ P_2[\underline{v}] + a^2P_{21}[\underline{u}_1,\underline{v}] + a^3P_{31}[\underline{u}_1,\underline{v}] + a^4P_{41}[\underline{u}_1,\underline{v}]\} &= \\ = - a^4P_2[\underline{v}_2] + O(a^5) \ , \end{aligned} \tag{3.6.37}$$

where $\underline{v}_2$ is the unique solution of (3.6.36).

We refrain from giving a more explicit result for the minimum (3.6.37), which would be available if full use had been made of (3.6.34) and (3.6.35), because we have already disregarded in (3.6.25) the explicit form of terms of the fifth and higher orders in the amplitude a of the buckling mode. Hence no useful purpose would be served by a more accurate evaluation of the minimum (3.6.37). It appears in fact that the terms $a^3P_{31}[\underline{u}_1,\underline{v}]$ and $a^4P_{41}[\underline{u}_1,\underline{v}]$ could already have been omitted from (3.6.25) and (3.6.26) without any detrimental effect on our result (3.6.37). The question therefore arises whether our analysis, leading from (3.6.26) to (3.6.37), is perhaps unnecessarily complicated by the retention of the two terms under discussion. We did in fact conjecture already at the start of

our analysis that only "small" deviations from a multiple of the critical buckling mode are likely to be significant in the discussion of stability. It was for this reason that an arbitrary kinematically admissible displacement field $\underline{u}$ was immediately written in the form (3.6.1). Our surmise that we may most likely confine our attention to "small" deviations $\underline{v}$ from a multiple of the buckling mode, was also supported by the appearance in (3.6.25), in addition to a quadratic term in $\underline{v}$, of only linear terms in $\underline{v}$ multiplied by a quadratic factor in a or a higher order factor in a. The most critical part of $\underline{v}$ was therefore likely to be of order a^2, a conjecture that is confirmed by our analysis. Nevertheless we emphasize that all these conjectures are in need of adequate proof. Such a proof can only be obtained from the analysis which we have given, or from a similar discussion. It does not seem possible to reject the terms $a^3P_{31}[\underline{u}_1,\underline{v}]$ and $a^4P_{41}[\underline{u}_1,\underline{v}]$ in (3.6.25) straight away without exposing the entire argument to legitimate objections.

We return now to a final discussion of (3.6.25). Remembering that the minimum values of (3.6.27) may be obtained from the result (3.6.37) for the minimum of (3.6.26) by dividing by $(1\mp\varepsilon)$, we have the inequalities

$$P[a\underline{u}_1+\underline{v}] \geqq a^4\{P_4[\underline{u}_1] - \frac{1}{1-\varepsilon} P_2[v_2]\} + O(a^5) \;, \quad (3.6.38)$$

$$\underset{(a=\text{const.})}{\text{Min.}} P[a\underline{u}_1+\underline{v}] \leqq a^4\{P_4[\underline{u}_1] - \frac{1}{1+\varepsilon} P_2[v_2]\} + O(a^5) \;, \quad (3.6.39)$$

where ε is a positive number that may be made as small as we please by selecting the positive constants $A,\overline{g},\overline{g}',\overline{g}'',\ldots$ in (3.6.3) sufficiently small. Introducing the constant (cf (3.6.34))

$$A_4 = P_4[\underline{u}_1] - P_2[\underline{v}_2] =$$
$$= P_4[\underline{u}_1] + P_{21}[\underline{u}_1,\underline{v}_2] + P_2[\underline{v}_2] \ , \qquad (3.6.40)$$

we obtain from (3.6.39) the <u>necessary condition for stability of the critical state of neutral equilibrium</u>

$$A_4 \geqq 0 \ , \qquad (3.6.41)$$

and from (3.6.38) the <u>sufficient condition for stability</u>

$$A_4 > 0 \ . \qquad (3.6.42)$$

These statements are justified by the fact that we may always make ε so small that the coefficients of a^4 in (3.6.38) and (3.6.39) have the <u>same sign</u> as A_4. We may then reduce the bound A on the modulus of a still further, if necessary, in order to ensure the domination of the first term in (3.6.38), (3.6.39) over the second term of order a^5. It will be clear that no decision is obtained as yet in the ambiguous case $A_4=0$. A continued investigation is required, if it is desired to obtain a decision also in this case.

It will be noticed that the necessary condition (3.6.41) requires <u>more</u> than a positive definite fourth variation $P_4[\underline{u}_1]$. It will be remembered from the discussion in section 3.2 that in the case $P_3[\underline{u}_1]=0$, $P_4[\underline{u}_1] > 0$ no further necessary conditions could be derived from (3.2.4). Condition (3.6.41) therefore contains the proof of our previous statement in section 3.2 that equilibrium may very well be unstable in spite of the satisfaction of (3.2.4) for every kinematically admissible displacement field.

The foregoing derivation and proof of the stability conditions (3.6.41) and (3.6.42) is based on an auxiliary quadratic functional $T_2[\underline{u}]$ whose typical integrand is a positive definite quadratic form in <u>all</u> arguments appearing in the typical integrand of the energy increase functional $P[\underline{u}]$ defined by (3.1.1). The discussion in section 3.4, however, enables us to relax the restrictions imposed on the auxiliary functional. Our entire argument remains valid, if we replace $T_2[\underline{u}]$ by any positive definite quadratic functional $\overset{*}{T}_2[\underline{u}]$, whose integrands are not necessarily positive definite, <u>provided that we still assume the existence of a solution to the original minimum problems (3.5.1) and (3.5.6), (3.5.7).</u> The solution

$$\overset{*}{\omega}_1 = \text{Min.}\ \frac{P_2[\underline{u}]}{\overset{*}{T}_2[\underline{u}]} = 0 \ , \qquad (3.6.43)$$

attained for the displacement field $\overset{*}{\underline{u}}_1$, then also ensures a <u>zero</u> solution of (3.5.1) for the <u>same displacement field</u> $\underline{u}_1 \equiv \overset{*}{\underline{u}}_1$. Moreover, a <u>positive solution</u> $\overset{*}{\omega}_2 > 0$ of the minimum problem

$$\overset{*}{\omega}_2 = \text{Min.}\ \frac{P_2[\underline{u}]}{\overset{*}{T}_2[\underline{u}]} \ , \qquad (3.6.44)$$

under the condition

$$\overset{*}{T}_{11}[\underline{u}_1,\underline{u}] = 0 \ , \qquad (3.6.45)$$

also ensures a <u>positive solution</u> $\omega_2 > 0$ of the minimum problem (3.5.6), (3.5.7), and vice versa.

Replacing (3.6.1) by

$$\underline{u} = a^* \underline{u}_1 + \underline{v}^* \text{ , where } T^*_{11}[\underline{u}_1, \underline{v}^*] = 0 \text{ ,} \qquad (3.6.46)$$

all steps in the previous analysis may be retraced up to the discussion of the remainder (3.6.19). Here we obtain with our <u>original</u> functional $T_2[\underline{u}]$ the estimate

$$| R_2[\underline{v}^*] | \leqq \varepsilon^{*\prime} T_2[\underline{v}^*] \text{ ,} \qquad (3.6.47)$$

similar to (3.6.23), and from our original inequality (3.6.2)

$$| R_2[\underline{v}^*] | \leqq \varepsilon^* P_2[\underline{v}^*] \text{ ,} \qquad (3.6.48)$$

where $\varepsilon^* = \varepsilon^{*\prime}/\omega_2$ is a positive number as small as we please. From here we may again retrace the steps of our previous analysis. Our final result is now obtained in terms of a constant A^*_4 , defined by

$$A^*_4 = P_4[\underline{u}_1] - P_2[\underline{v}^*_2] \text{ ,} \qquad (3.6.49)$$

where $\underline{v}^*_2$ is the unique solution of the equation

$$P_{11}[\underline{v}^*, \underline{\eta}^*] + P_{21}[\underline{u}_1, \underline{\eta}^*] = 0 \qquad (3.6.50)$$

which satisfies the orthogonality condition $T^*_{11}[\underline{u}_1, \underline{v}^*] = 0$. The restriction (3.6.28) on the otherwise arbitrary kinematically admissible displacement field $\underline{\eta}$ is here of course replaced by the analogous condition for $\underline{\eta}^*$

$$T^*_{11}[\underline{u}_1, \underline{\eta}^*] = 0 \text{ .} \qquad (3.6.51)$$

<u>The necessary condition for stability is now</u> $A^*_4 \geqq 0$, <u>a sufficient condition is</u> $A^*_4 > 0$.

It goes without saying that the verdict on stability must be independent of the particular choice of auxiliary positive definite functional $T_2[\underline{u}]$. Hence we must have the <u>identity</u>

$$A_4^* = A_4 \ . \tag{3.6.52}$$

An analytical proof of this identity is not immediately obvious. The different restrictions (3.6.28) and (3.6.51), imposed on $\underline{\eta}$ and $\underline{\eta}^*$, prevent an immediate comparison of the solution $\underline{v}_2$ of (3.6.36) and the solution $\underline{v}_2^*$ of (3.6.50). Removal of these restrictions is also desirable for other reasons. We shall prove therefore that these equations remain valid, if $\underline{\eta}$ and $\underline{\eta}^*$ are replaced by an arbitrary kinematically admissible displacement field $\underline{\zeta}$ which is <u>not</u> subject to any orthogonality condition of type (3.6.28) or (3.6.51). It will be sufficient to confine our attention to (3.6.36) because the argument in the case of equation (3.6.50) is entirely similar. Writing the arbitrary kinematically admissible displacement field $\underline{\zeta}$ in the form (cf.(3.4.34))

$$\underline{\zeta} = \frac{1}{2}\,\underline{u}_1 T_{11}[\underline{u}_1,\underline{\zeta}] + \underline{\eta} = t\underline{u}_1+\underline{\eta} \ , \tag{3.6.53}$$

where $\underline{\eta}$ satisfies (3.6.28), we may evaluate the left-hand member of (3.6.36) for its solution $\underline{v}_2$, and after replacement of $\underline{\eta}$ by $\underline{\zeta}$

$$P_{11}[\underline{v}_2,\underline{\zeta}] + P_{21}[\underline{u}_1,\underline{\zeta}] =$$

$$= t\{P_{11}[\underline{v}_2,\underline{u}_1] + P_{21}[\underline{u}_1,\underline{u}_1]\} + P_{11}[\underline{v}_2,\underline{\eta}] + P_{21}[\underline{u}_1,\underline{\eta}] \ . \tag{3.6.54}$$

The first term within the braces vanishes because $\underline{u}_1$ is the buckling mode (cf.(3.5.3)), and the second term within these braces

is also zero on account of (3.6.22)

$$P_{21}[\underline{u}_1,\underline{u}_1] = 3P_3[\underline{u}_1] = 0 \tag{3.6.55}$$

The last two terms cancel because $\underline{v}_2$ is the solution of (3.6.36). Hence we have proved that $\underline{v}_2$ is also a solution of the modified variational equation

$$P_{11}[\underline{v},\underline{\zeta}] + P_{21}[\underline{u}_1,\underline{\zeta}] = 0 \ , \tag{3.6.56}$$

holding for <u>every</u> kinematically admissible displacement field $\underline{\zeta}$. In fact, $\underline{v}_2$ is the <u>unique</u> solution of (3.6.56) which also satisfies the <u>orthogonality relation</u>

$$T_{11}[\underline{u}_1,\underline{v}_2] = 0 \ . \tag{3.6.57}$$

Likewise, $\underline{v}_2^*$ is the <u>unique</u> solution of the same equation (3.6.56) which also satisfies the <u>orthogonality relation</u>

$$T_{11}^*[\underline{u}_1,\underline{v}_2^*] = 0 \ . \tag{3.6.58}$$

Equation (3.5.56), satisfied by both $\underline{v}_2$ and $\underline{v}_2^*$, enables us to compare these solutions. Their difference must be a solution of the equation

$$P_{11}[\underline{v},\underline{\zeta}] = 0 \ . \tag{3.6.59}$$

Since the critical buckling mode is simple, the general solution for the difference of $\underline{v}_2^*$ and $\underline{v}_2$ is

$$\underline{v}_2^*-\underline{v}_2 = b\underline{u}_1 \ , \tag{3.6.60}$$

where b is an arbitrary constant. Hence we have

$$P_2[\underline{v}_2^*] = P_2[\underline{v}_2] + bP_{11}[\underline{v}_2,\underline{u}_1] + b^2P_2[\underline{u}_1] = P_2[\underline{v}_2] \ , \tag{3.6.61}$$

if again appropriate use is made of (3.5.3) and of (3.5.2). Substitution of (3.6.61) into (3.6.49) completes our proof of (3.6.52).

A second reason why the modified variational equation (3.6.56) is to be preferred to the original equation (3.6.36) or (3.6.50) is that the modified equation is equivalent to a set of differential equations and boundary conditions. These differential equations and boundary conditions are nearly always indispensable in the solution of any particular problem of stability in a critical case of neutral equilibrium. Their derivation from (3.6.56) proceeds along the same lines as the derivation of the equations of neutral equilibrium from the variational equation (3.5.3). The fact that the first term in (3.6.56) and the left-hand member in (3.5.3) are identical implies that the unknown displacement field $\underline{v}$ and its derivatives appear in our present equations in exactly the same form as $\underline{u}$ and its derivatives appear in the equations of neutral equilibrium. The only difference is that the present equations are no longer homogeneous.

A curious difficulty is apparently raised here by the general theory of linear differential equations. The homogeneous equations have a non-vanishing solution in the form of the critical buckling mode because they are identical with the equations of neutral equilibrium. In order that a solution of the non-homogeneous equations may exist, their prescribed right-hand members have then to satisfy a certain necessary condition in the

form of an integral relation.[*] The question then arises whether this relation is actually satisfied by the right-hand members of the equations resulting from the variational equation (3.6.56).

The answer to the latter question is fortunately affirmative, and the difficulty raised by the general theory of linear differential equations is therefore no more than apparent. In the interest of future applications, it will be convenient to prove this statement in a slightly more general context than is actually needed for the purpose at hand. Our variational equation (3.6.56) is a particular case of the more general equation

$$P_{11}[\underline{v},\underline{\zeta}] + \overline{M}_1[\underline{\zeta}] = 0 \,, \qquad (3.6.62)$$

where $\overline{M}_1[\underline{\zeta}]$ is a linear functional of the arbitrary kinematically admissible displacement field $\underline{\zeta}$. The associated non-homogeneous linear differential equations and boundary conditions admit a solution, only if the right-hand members satisfy a single integral relation. Since the right-hand members result from the term $\overline{M}_1[\underline{\zeta}]$ in (3.6.62), it follows that a solution exists, only if the functional $\overline{M}_1[\underline{\zeta}]$ obeys an equivalent condition. The explicit form of the latter condition, however, is obtained immediately from (3.6.62) by observing that this equation must also hold for $\underline{\zeta}=\underline{u}_1$. The first term in (3.6.62) vanishes in this case, and it follows

[*] This necessary condition is usually called an orthogonality condition in the theory of differential equations. We shall avoid this terminology in order to prevent the danger of confusion with our concept of orthogonality based on the auxiliary quadratic functional $T_2[\underline{u}]$.

that the linear functional $\overline{M}_1[\underline{\zeta}]$ must satisfy the condition

$$\overline{M}_1[\underline{u}_1] = 0 , \tag{3.6.63}$$

in order that the variational equation (3.6.62) may have a solution. Condition (3.6.63) must therefore be the equivalent of the integral relation to be satisfied by the right-hand members of the differential equation and boundary conditions. It is easily verified by means of (3.6.55) that in the particular case under consideration,

$$\overline{M}_1[\underline{\zeta}] = P_{21}[\underline{u}_1, \underline{\zeta}] , \tag{3.6.64}$$

condition (3.6.63) is indeed satisfied. It follows that the right-hand members of the differential equations and boundary conditions associated with (3.6.56) actually satisfy the necessary condition in question for the existence of a solution of the non-homogeneous equations.

There is a deeper reason, apart from the vanishing of $P_3[\underline{u}_1]$ in the particular case under consideration, why the non-homogeneous variational equations of the type (3.6.62), which we shall encounter in various special forms in the sequel, <u>always</u> satisfy the integral relation which is a necessary condition for the existence of a solution of the associated non-homogeneous differential equations and boundary conditions. It may therefore be worthwhile to pursue the discussion a little further, even if this is not strictly necessary for our immediate purposes.

Variational equations of type (3.6.62) always enter our discussion as the necessary and sufficient condition for the

existence of a solution of a minimum problem

$$P_2[\underline{v}] + M_1[\underline{v}] = \text{Min.} \tag{3.6.65}$$

under the side condition

$$T_{11}[\underline{u}_1,\underline{v}] = 0 \ , \tag{3.6.66}$$

where $M_1[\underline{v}]$ is some <u>linear</u> functional of $\underline{v}$. We shall assume that its typical integrand depends only (and that of course in linear form) on the displacement components and their derivatives which also occur in the energy increase functional (3.1.1) and in $T_2[\underline{u}]$. It is easily verified that the minimum problem (3.6.65), (3.6.66) is <u>properly formulated</u> in the sense that the existence of a lower bound of (3.6.65) is ensured. In fact, we have

$$P_2[\underline{v}] + M_1[\underline{v}] \geqq \omega_2 T_2[\underline{v}] + M_1[\underline{v}] \ , \tag{3.6.67}$$

and the typical integrand of the right-hand member in (3.6.67) is obviously bounded below. Let $\underline{v}'$ denote the solution of (3.6.65), whose existence will now again be assumed, and let $\underline{\eta}$ again denote an arbitrary kinematically admissible displacement field which satisfies the orthogonality condition (3.6.28). The minimum property of $\underline{v}'$ is then expressed by the inequality of well-known type

$$\varepsilon\{P_{11}[\underline{v}'\underline{\eta}] + M_1[\underline{\eta}]\} + \varepsilon^2 P_2[\underline{\eta}] \geqq 0 \ , \tag{3.6.68}$$

where ε is an arbitrary constant. The necessary and sufficient condition resulting from this inequality is the equation

$$P_{11}[\underline{v},\underline{\eta}] + M_1[\underline{\eta}] = 0 \ , \tag{3.6.69}$$

satisfied by $\underline{v}=\underline{v}'$ for <u>every</u> displacement field $\underline{\eta}$ subject to restriction (3.6.28). In order to obtain from (3.6.69) a variational equation in which the arbitrary kinematically admissible displacement field $\underline{\zeta}$ is <u>not</u> subject to the restriction (3.6.28), we substitute (3.6.53) in the left-hand member of (3.6.69), and evaluate the result. We obtain

$$P_{11}[\underline{v},\underline{\zeta}] + M_1[\underline{\zeta}] = \{P_{11}[\underline{v},\underline{u}_1] + M_1[\underline{u}_1]\} \frac{1}{2} T_{11}[\underline{u}_1,\underline{\zeta}] + $$

$$+ P_{11}[\underline{v},\underline{\eta}] + M_1[\underline{\eta}] = \frac{1}{2} M_1[\underline{u}_1]T_{11}[\underline{u}_1,\underline{\zeta}] \ , \tag{3.6.70}$$

where we have used (3.5.3) and (3.6.69). Writing this result in the form

$$P_{11}[\underline{v},\underline{\zeta}] + M_1[\underline{\zeta}] - \frac{1}{2} M_1[\underline{u}_1]T_{11}[\underline{u}_1,\underline{\zeta}] = 0 \ , \tag{3.6.71}$$

we have obtained an equation of type (3.6.62), in which

$$\overline{M}_1[\underline{\zeta}] = M_1[\underline{\zeta}] - \frac{1}{2} M_1[\underline{u}_1]T_{11}[\underline{u}_1,\underline{\zeta}] \tag{3.6.72}$$

<u>automatically satisfies condition (3.6.63)</u>. We need not fear therefore that the right-hand members of the differential equations and boundary conditions associated with (3.6.71) would violate the integral relation which is a necessary condition for the existence of a solution. The satisfaction of this condition is in fact <u>a consequence of the proper formulation of minimum problem (3.6.65), (3.6.66)</u>.

Finally, a word of caution may be in order here to prevent the misunderstanding that the foregoing discussion proves

the existence of a solution of minimum problems of type (3.6.65), (3.6.66), their associated variational equations of type (3.6.62) or the corresponding differential equations and boundary conditions. The quite difficult issue of existence of solutions is far beyond the scope of our discussion. Our purpose has merely been to show that <u>the well-known necessary conditions for the existence of a solution of non-homogeneous differential equations and boundary conditions, are always satisfied in our equations resulting from properly formulated minimum problems</u>.

The actual <u>application of the stability criterion</u>, specified by the constant A_4 (3.6.40), is extremely simple in the case of our <u>first example</u>, <u>the bar under compressive end loads</u> (fig. 3.1). A critical case of neutral equilibrium occurs, if and only if the compressive load N equals the Euler load

$$N = N_1 = \frac{\pi^2 B}{\ell^2} \ . \tag{3.6.73}$$

The corresponding buckling mode (3.5.20) for k=1 may be normalized here by taking the maximum deflection equal to unity

$$w_1 = \sin \pi x/\ell \ . \tag{3.6.74}$$

The third variation $P_3[w]$ vanishes <u>identically</u> in this example (cf. (3.2.12)). Hence the second term in the variational equation (3.6.56) is also identically zero. The general solution of this equation is a multiple of the buckling mode (3.6.74), and it follows that $\underline{v}_2 \equiv 0$ in view of the required orthogonality to the buckling mode. Hence the constant A_4 is in this case supplied immediately by the fourth variation $P_4[w]$, evaluated for the

buckling mode (3.6.74). We obtain

$$A_4 = P_4[w_1] = \frac{\pi^4}{2\ell^4}\int_0^\ell [\frac{\pi^2 B}{\ell^2} \sin^2 \frac{\pi x}{\ell} \cos^2 \frac{\pi x}{\ell} - \frac{1}{4} N_1 \cos^4 \frac{\pi x}{\ell}]dx =$$

$$= \frac{\pi^6 B}{64\ell^5} . \tag{3.6.75}$$

This constant A_4 is positive, and the critical case of neutral equilibrium at the Euler load is therefore stable.

The second example, the three-dimensional elastic body, is far more complicated. No general solution of the equations of neutral equilibrium is available in this case. If the buckling mode in a critical case of neutral equilibrium is again denoted by $\underline{u}_1$, the displacement components by u_{1i} $(i=1,2,3)$, it would of course be possible to derive the differential equations corresponding to the variational equation (3.6.56) along the same lines as the derivation of the equations of neutral equilibrium in section 3.5. The resulting equations are extremely complicated, and little additional insight is gained from them. Let it therefore suffice to give the appropriate expression for the second term in (3.6.56). The interested reader may then derive the associated explicit differential equations and boundary conditions. From (3.2.18) we obtain in this example

$$P_{21}[\underline{u}_1,\underline{\zeta}] = \int_V G[(u_{1i,j}+u_{1j,i})u_{1h,i}\zeta_{h,j} + \frac{1}{2} u_{1h,i}u_{1h,j}(\zeta_{i,j}+\zeta_{j,i}) +$$

$$+ \frac{2\nu}{1-2\nu} u_{1k,k}u_{1h,i}\zeta_{h,i} + \frac{\nu}{1-2\nu} u_{1h,i}u_{1h,i}\zeta_{k,k}]dv . \tag{3.6.76}$$

A final remark seems appropriate before closing the present section. It should be noted that the detailed investigation of stability in a critical case of neutral equilibrium is necessary only in cases which ought perhaps to be classified as "exceptional" from a mathematical point of view. The "general" case in a mathematical sense would, of course, be that the third variation $P_3[\underline{u}]$ does not vanish for the buckling mode $\underline{u}_1$, that is

$$P_3[\underline{u}_1] \neq 0 \ . \tag{3.6.77}$$

Instability of the critical state of neutral equilibrium is then immediately obvious from the necessary condition for stability (3.2.7) or (3.6.22), and any further investigation is superfluous. Nevertheless, many problems do exist in which this necessary condition (3.2.7) happens to be satisfied, and the continued analysis of the present section is indeed required in such cases. The vanishing of $P_3[\underline{u}_1]$ is usually caused by certain symmetry properties of the structure and/or its buckling mode, which overrule the prediction that (3.6.77) is the general case in a mathematical sense.

3.7 The indecisive case.

In the preceding section we have derived necessary and sufficient conditions for the stability of the fundamental state in a critical case of neutral equilibrium characterized by a simple critical buckling mode $\underline{u}_1$. The discussion of the case of multiple buckling modes is postponed until the next section. The necessary and sufficient conditions were obtained in the form of the inequalities $A_4 \geqq 0$ and $A_4 > 0$ respectively, where A_4 is a constant defined by (3.6.40). We shall now examine how the investigation may be continued in the indecisive case $A_4=0$.

At a first glance it may appear sufficient for this purpose to refine the accuracy of the estimates (3.6.9) and (3.6.37) by taking into account explicitly the terms of order a^5. It would then be overlooked, however, that the remainder (3.6.19) in (3.6.20) also involves a term of order a^5, if the solution (3.6.35) is substituted into (3.6.19). A more careful analysis is therefore required. We may of course expect that the decision on stability will be given by a displacement field close to

$$\underline{u} = a\underline{u}_1+a^2\underline{v}_2 \ , \tag{3.7.1}$$

which barely fails to yield the required verdict. Hence it will obviously be expedient to write an arbitrary kinematically admissible displacement field in the form

$$\underline{u} = a\underline{u}_1+a^2\underline{v}_2+\underline{w} \ , \quad \text{where } T_{11}[\underline{u}_1,\underline{w}] = 0 \ . \tag{3.7.2}$$

We need a convenient notation for the expansion of a functional $S_m[\underline{u}]$, whose integrand is a homogeneous polynomial of degree m in the displacement components u_i and their derivatives, similar to (3.4.6), but now for the sum of three displacement fields. We write

$$S_m[\underline{u}+\underline{v}+\underline{w}] = \sum_{(p+q+r=m)} S_{pqr}[\underline{u},\underline{v},\underline{w}] , \qquad (3.7.3)$$

where the typical integrand of $S_{pqr}[\underline{u},\underline{v},\underline{w}]$ contains all terms of degree p in $\underline{u}$, of degree q in $\underline{v}$, and of degree r in $\underline{w}$. We note the six symmetry relations

$$S_{pqr}[\underline{u},\underline{v},\underline{w}] = S_{qpr}[\underline{v},\underline{u},\underline{w}] = S_{rpq}[\underline{w},\underline{u},\underline{v}] =$$
$$S_{prq}[\underline{u},\underline{w},\underline{v}] = S_{qrp}[\underline{v},\underline{w},\underline{u}] = S_{rqp}[\underline{w},\underline{v},\underline{u}] , \qquad (3.7.4)$$

and the identities similar to (3.4.7)

$$S_{oqr}[\underline{u},\underline{v},\underline{w}] = S_{qr}[\underline{v},\underline{w}],\ S_{oom}[\underline{u},\underline{v},\underline{w}] = S_m[\underline{w}] , \qquad (3.7.5)$$
$$S_{pqr}[\underline{u},\underline{u},\underline{w}] = \frac{(p+q)!}{p!q!} S_{(p+q)r}[\underline{u},\underline{w}],\ S_{pqr}[\underline{u},\underline{u},\underline{u}] = \frac{m!}{p!q!r!} S_m[u]. \qquad (3.7.6)$$

The arguments y^λ of the typical integrand $F(\underline{y})$ of $P[\underline{u}]$ are written in a form corresponding to (3.7.2) and similar to (3.6.6)

$$y^\lambda = ay_1^\lambda+a^2z_2^\lambda+s^\lambda , \quad \text{or } \underline{y} = a\underline{y}_1+a^2\underline{z}_2+\underline{s} , \qquad (3.7.7)$$

where $\underline{z}_2$ now represents the arguments corresponding to $\underline{v}_2$. The Taylor-expansion of the typical integrand is now written in the form

$$F(a\underline{y}_1+a^2\underline{z}_2+\underline{s}) = F(a\underline{y}_1+a^2\underline{z}_2) + s^\lambda F_{,\lambda}(a\underline{y}_1+a^2\underline{z}_2) +$$

$$+ \frac{1}{2} s^\lambda s^\mu F_{,\lambda\mu}(a\underline{y}_1+a^2\underline{z}_2) + \frac{1}{6} s^\lambda s^\mu s^\nu F_{,\lambda\mu\nu}(a\underline{y}_1+a^2\underline{z}_2+\underline{\bar{s}}) , \quad (3.7.8)$$

where the arguments in the third order derivative are given by

$$ay_1^\lambda+a^2z_2^\lambda+\bar{s}^\lambda = ay_1^\lambda+a^2z_2^\lambda+\theta^\lambda s^\lambda , \quad (\lambda \text{ not summed; } 0 \leqq \theta^\lambda \leqq 1) . \quad (3.7.9)$$

The second derivatives in (3.7.8) are rewritten as

$$F_{,\lambda\mu}(a\underline{y}_1+a^2\underline{z}_2) = F_{,\lambda\mu}(0) + (ay_1^\nu+a^2z_2^\nu)F_{,\lambda\mu\nu}(\theta a\underline{y}_1+\theta a^2\underline{z}_2) , \quad (0 \leqq \theta \leqq 1) . \quad (3.7.10)$$

We may now proceed to identify the various terms after the collection of all integrals. We obtain the symbolic relations

$$\begin{aligned} F(a\underline{y}_1+a^2\underline{z}_2) \Longrightarrow\ & a^3P_3[\underline{u}_1] + a^4\{P_4[\underline{u}_1] + P_{21}[\underline{u}_1,\underline{v}_2] + P_2[\underline{v}_2]\} + \\ & + a^5\{P_5[\underline{u}_1] + P_{31}[\underline{u}_1,\underline{v}_2] + P_{12}[\underline{u}_1,\underline{v}_2]\} + \\ & + a^6\{P_6[\underline{u}_1] + P_{41}[\underline{u}_1,\underline{v}_2] + P_{22}[\underline{u}_1,\underline{v}_2] + P_3[\underline{v}_2]\} + \\ & + O(a^7) , \end{aligned} \quad (3.7.11)$$

$$\begin{aligned} s^\lambda F_{,\lambda}(a\underline{y}_1+a^2\underline{z}_2) \Longrightarrow\ & a^2\{P_{21}[\underline{u}_1,\underline{w}] + P_{11}[\underline{v}_2,\underline{w}]\} + \\ & + a^3\{P_{31}[\underline{u}_1,\underline{w}] + P_{111}[\underline{u}_1,\underline{v}_2,\underline{w}]\} + \\ & + a^4\{P_{41}[\underline{u}_1,\underline{w}] + P_{211}[\underline{u}_1,\underline{v}_2,\underline{w}] + P_{21}[\underline{v}_2,\underline{w}]\} + \\ & + a^5\{P_{51}[\underline{u}_1,\underline{w}] + P_{311}[\underline{u}_1,\underline{v}_2,\underline{w}] + P_{121}[\underline{u}_1,\underline{v}_2,\underline{w}]\} + \\ & + a^6\{P_{61}[\underline{u}_1,\underline{w}] + P_{411}[\underline{u}_1,\underline{v}_2,\underline{w}] + P_{221}[\underline{u}_1,\underline{v}_2,\underline{w}] + P_{31}[\underline{v}_2,\underline{w}]\} + \\ & + O(a^7) , \end{aligned} \quad (3.7.12)$$

$$\frac{1}{2}\, s^\lambda s^\mu F_{,\lambda\mu}(0) \Longrightarrow P_2[\underline{w}] \ , \qquad (3.7.13)$$

For the remainder of (3.7.8) we have the relation

$$\overline{\overline{A}}_{\lambda\mu} s^\lambda s^\mu \Longrightarrow \overline{R}_2[\underline{w}] \ , \qquad (3.7.14)$$

where $\overline{\overline{A}}_{\lambda\mu}$ is defined by

$$\overline{\overline{A}}_{\lambda\mu} = \frac{1}{6}\, s^\nu F_{,\lambda\mu\nu}(a\underline{y}_1+a^2\underline{z}_2+\underline{\overline{s}}) +$$

$$+ \frac{1}{2}\,(ay_1^\nu+a^2z_2^\nu)F_{,\lambda\mu\nu}(\Theta a\underline{y}_1+\Theta a^2\underline{z}_2) \ . \qquad (3.7.15)$$

Before we write down our resulting expression for the energy increase in the displacement field (3.7.2) from the fundamental state to some state II, it is convenient to make use of the simplifications available from our previous investigation. We observe that the first term and the second term in (3.7.11) vanish by virtue of (3.6.22) and the fact that in our indecisive case

$$A_4 = P_4[\underline{u}_1] + P_{21}[\underline{u}_1,\underline{v}_2] + P_2[\underline{v}_2] = 0 \ . \qquad (3.7.16)$$

Restricting our attention first to a displacement field (3.7.1), i.e. to $\underline{w} \equiv 0$, the <u>entire</u> energy increase is given by (3.7.11). A necessary condition for stability is then that the lowest power of a in (3.7.11) is even and has a non-negative coefficient. Hence we obtain the <u>necessary conditions for stability in the indecisive case</u> $A_4=0$

$$P_5[\underline{u}_1] + P_{31}[\underline{u}_1,\underline{v}_2] + P_{12}[\underline{u}_1,\underline{v}_2] = 0 \ , \qquad (3.7.17)$$

$$P_6[\underline{u}_1] + P_{41}[\underline{u}_1,\underline{v}_2] + P_{22}[\underline{u}_1,\underline{v}_2] + P_3[\underline{v}_2] \geqq 0 \ . \qquad (3.7.18)$$

Henceforward we shall assume that these necessary conditions are satisfied.

If we now consider (3.7.12), we observe that the first term vanishes identically because $\underline{v}_2$ is the solution of (3.6.36). Moreover, our previous experience with the terms which are linear in $\underline{v}$ in (3.6.20) indicates that only the term with the lowest power of a as a factor is actually significant. We shall profit from this experience by omitting in (3.7.12) all terms except the first non-vanishing term. We emphasize, however, that this omission lacks a proper justification at this stage. A rigorous analysis would require retention of all terms in (3.7.12), and it would then appear a posteriori that only the term with a^3 as a factor is significant. It may be left to the reader to complete the proof in this sense.

Our resulting simplified expression for the energy increase (3.1.1) in the displacement field (3.7.2) reads

$$\begin{aligned} P[a\underline{u}_1+a^2\underline{v}_2+\underline{w}] = {} & a^6\{P_6[\underline{u}_1] + P_{41}[\underline{u}_1,\underline{v}_2] + P_{22}[\underline{u}_1,\underline{v}_2] + P_3[\underline{v}_2]\} + \\ & + a^3\{P_{31}[\underline{u}_1,\underline{w}] + P_{111}[\underline{u}_1,\underline{v}_2,\underline{w}]\} + \\ & + P_2[\underline{w}] + \overline{R}_2[\underline{w}] + O(a^7) \ . \qquad (3.7.19) \end{aligned}$$

The further analysis is the complete analogue of the investigation of (3.6.20) in the preceding section, and it may again be left to the reader to fill in the details of the following discussion. It is first established that the remainder in (3.7.19) satisfies the inequality

$$|\bar{R}_2[\underline{w}]| \leqq \varepsilon P_2[\underline{w}] \;, \qquad (3.7.20)$$

where ε is a positive number as small as we please.

Let $\underline{w}=a^3\underline{w}_3$ denote the solution of the minimum problem

$$P_2[\underline{w}] + a^3\{P_{31}[\underline{u}_1,\underline{w}] + P_{111}[\underline{u}_1,\underline{v}_2,\underline{w}]\} = \text{Min.}, \qquad (3.7.21)$$

where the displacement field $\underline{w}$ is restricted by the side condition

$$T_{11}[\underline{u}_1,\underline{w}] = 0 \;. \qquad (3.7.22)$$

The displacement field $\underline{w}_3$ is a solution of the variational equation

$$P_{11}[\underline{w},\underline{\eta}] + P_{31}[\underline{u}_1,\underline{\eta}] + P_{111}[\underline{u}_1,\underline{v}_2,\underline{\eta}] = 0 \;, \qquad (3.7.23)$$

holding for every kinematically admissible displacement field $\underline{\eta}$ subject to the orthogonality restriction (3.6.28). The solution $\underline{w}_3$ is the unique solution of (3.7.23) which also satisfies the orthogonality condition (3.7.22). The variational equation (3.7.23) may again be replaced by a similar equation in terms of an arbitrary kinematically admissible displacement field $\underline{\zeta}$ which is <u>not</u> subject to the orthogonality restriction (3.6.28) on $\underline{\eta}$. In view of the identity

$$P_{31}[\underline{u}_1,\underline{u}_1] + P_{111}[\underline{u}_1,\underline{v}_2,\underline{u}_1] = 4P_4[\underline{u}_1] + 2P_{21}[\underline{u}_1,\underline{v}_2] =$$

$$= 4\{P_4[\underline{u}_1] - P_2[\underline{v}_2]\} = 0 \;, \qquad (3.7.24)$$

resulting from (3.6.40) and $A_4=0$, the latter equation has exactly the same form as (3.7.23)

$$P_{11}[\underline{w},\underline{\zeta}] + P_{31}[\underline{u}_1,\underline{\zeta}] + P_{111}[\underline{u}_1,\underline{v}_2,\underline{\zeta}] = 0 \qquad (3.7.25)$$

Introducing the constant A_6, defined by

$$A_6 = P_6[\underline{u}_1] + P_{41}[\underline{u}_1, \underline{v}_2] + P_{22}[\underline{u}_1, \underline{v}_2] + P_3[\underline{v}_2] - P_2[\underline{w}_3] , \tag{3.7.26}$$

a further <u>necessary</u> condition for stability in the indecisive case $A_4=0$ is

$$A_6 \geqq 0 ; \tag{3.7.27}$$

and a <u>sufficient</u> condition for stability is

$$A_6 > 0 . \tag{3.7.28}$$

No decision is obtained once again, if the constant A_6 equals zero, and the analysis has to be continued anew in this indecisive case of higher order. The procedure to be followed then will be clear from the foregoing analysis. It does not present any new difficulties, although the energy expression for a displacement field

$$\underline{u} = a\underline{u}_1 + a^2\underline{v}_2 + a^3\underline{w}_3 + \overline{\underline{w}} , \quad \text{where } T_{11}[\underline{u}_1, \overline{\underline{w}}] = 0 , \tag{3.7.29}$$

and the corresponding equations become increasingly more complex. We may omit a further discussion, the more so since the occurrence of this quite exceptional case seems highly unlikely in practical applications of the theory.

3.8 The case of multiple buckling modes.

The investigation of stability in a critical case of neutral equilibrium in sections 3.6 and 3.7 has been restricted to the case of a simple critical buckling mode. We shall now discuss the slightly modified analysis which is required in the case of multiple buckling modes. Let n denote the number of linearly independent critical buckling modes. The first n solutions in the sequence of minimum problems (3.4.26), (3.4.27) are then all zero, and the solution of the (n+1)-st problem is positive

$$\omega_1 = \omega_2 = \ldots = \omega_n = 0 \ , \ \ \omega_{(n+1)} > 0 \ . \qquad (3.8.1)$$

The buckling modes $\underline{u}_h$ (h=1,2,...n) are the minimizing displacement fields in the first n problems, normalized by the condition $T_2[\underline{u}_h] = 1$.

It is expedient here to represent an arbitrary kinematically admissible displacement field $\underline{u}$ by means of (3.4.23) in the form

$$\underline{u} = a_h\underline{u}_h + \underline{v} \ , \ \text{where } T_{11}[\underline{u}_h, \underline{v}] = 0 \ , \ h=1,2,\ldots n \qquad (3.8.2)$$

Here and in the remainder of this section we shall employ the summation convention for a repeated Latin subscript in the sense that such a repeated subscript implies summation from 1 to n, in the absence of an explicit statement to the contrary.

The arguments y^λ of the typical integrand $F(\underline{y})$ of $P[\underline{u}]$, defined in section 3.2, are written in a form corresponding to (3.8.2), and similar to (3.6.6)

$$y^\lambda = a_h y_h^\lambda + z^\lambda \ , \text{ or } \underline{y} = a_h \underline{y}_h + \underline{z} \ , \qquad (3.8.3)$$

where y_h^λ represents these arguments in the buckling mode $\underline{u}_h$. The analysis now follows again along the same lines as in section 3.6 with the single exception that $a\underline{u}_1$ and $a\underline{y}_1$ are replaced everywhere by the sums $a_h\underline{u}_h$ and $a_h\underline{y}_h$. The first important result is an expression for the energy increase (3.1.1), similar to (3.6.20)

$$P[a_h\underline{u}_h+\underline{v}] = P_3[a_h\underline{u}_h] + P_4[a_h\underline{u}_h] + P_{21}[a_h\underline{u}_h,\underline{v}] + P_{31}[a_h\underline{u}_h,\underline{v}] +$$
$$+ P_{41}[a_h\underline{u}_h,\underline{v}] + P_2[\underline{v}] + R_2[\underline{v}] + O(a^5) \ , \qquad (3.8.4)$$

where the symbol $O(a^5)$ stands for a term which tends to zero as a homogeneous polynomial of the fifth degree in the amplitudes a_h of the buckling modes, if these amplitudes themselves approach zero. An inequality similar to (3.6.24) may be proved for the term $R_2[\underline{v}]$, viz.

$$|R_2[\underline{v}]| \leqq \varepsilon P_2[\underline{v}] \ , \qquad (3.8.5)$$

where ε is again a positive number as small as we please. In the present case this inequality is based on the inequality for $\omega_{(n+1)} > 0$

$$P_2[\underline{v}] \geqq \omega_{(n+1)} T_2[\underline{v}] \ , \qquad (3.8.6)$$

which replaces (3.6.2).

It is convenient to derive immediately some <u>necessary</u> conditions for stability by considering a displacement field $\underline{u}$ which is a linear combination of the critical buckling modes, i.e. by taking $\underline{v} \equiv 0$ in (3.8.4). We must obviously have then

$$P_3[a_h \underline{u}_h] = 0 \; , \qquad (3.8.7)$$

$$P_4[a_h \underline{u}_h] \geqq 0 \; , \qquad (3.8.8)$$

for every choice of the amplitudes a_h. We introduce the constants*)

$$A_{hij} = \frac{1}{6} P_{111}[\underline{u}_h, \underline{u}_i, \underline{u}_j] \; , \quad B_{hijk} = \frac{1}{24} P_{1111}[\underline{u}_h, \underline{u}_i, \underline{u}_j, \underline{u}_k] \; , \qquad (3.8.9)$$

where the last functional is defined as the sum of all terms in the expansion of $P_4[\underline{u}]$ for $\underline{u}=\underline{u}_h+\underline{u}_i+\underline{u}_j+\underline{u}_k$, which are linear in all four constituents $\underline{u}_h, \underline{u}_i, \underline{u}_j$ and $\underline{u}_k$. Condition (3.8.7) is then expressed by the requirement that the cubic form

$$A_{hij} a_h a_i a_j = 0 \qquad (3.8.10)$$

must vanish <u>identically</u>. Hence we must have

$$A_{hij} = \frac{1}{6} P_{111}[\underline{u}_h, \underline{u}_i, \underline{u}_j] = 0 \; , \; h,i,j=1,2,\ldots n \; . \qquad (3.8.11)$$

Likewise, condition (3.8.8) requires that the <u>quartic form</u>

$$B_{hijk} a_h a_i a_j a_k \geqq 0 \qquad (3.8.12)$$

is <u>non-negative for every combination of amplitudes of the buckling modes</u>. Henceforward we shall assume that both conditions (3.8.11) and (3.8.12) are satisfied.

Again similar to (3.6.26) we now investigate the <u>functional of the displacement field $\underline{v}$</u> orthogonal to all buckling modes, defined by

---

*) We reserve the symbol A_{hijk} for later use.

$$P_2[\underline{v}] + P_{21}[a_h\underline{u}_h,\underline{v}] + P_{31}[a_h\underline{u}_h,\underline{v}] + P_{41}[a_h\underline{u}_h,\underline{v}] \,, \tag{3.8.13}$$

for prescribed arbitrary constant values of the amplitudes of the buckling modes. Let $\underline{v}'$ be the displacement field $\underline{v}$ which minimizes the functional (3.8.13). It is the unique solution, orthogonal to all critical buckling modes, of the variational equation

$$P_{11}[\underline{v},\underline{\eta}] + P_{21}[a_h\underline{u}_h,\underline{\eta}] + P_{31}[a_h\underline{u}_h,\underline{\eta}] + P_{41}[a_h\underline{u}_h,\underline{\eta}] = 0 \,, \tag{3.8.14}$$

holding for <u>every</u> kinematically admissible displacement field $\underline{\eta}$ which satisfies the requirement of orthogonality to all buckling modes

$$T_{11}[\underline{u}_h,\underline{\eta}] = 0 \,, \; h=1,2,\ldots n \,. \tag{3.8.15}$$

Equation (3.8.14) is again linear in $\underline{v}$. We write

$$P_{21}[a_h\underline{u}_h,\underline{\eta}] = \frac{1}{2}\, a_h a_i P_{111}[\underline{u}_h,\underline{u}_i,\underline{\eta}] \,. \tag{3.8.16}$$

The solution of (3.8.14) may then be written in the form

$$\underline{v}' = a_h a_i \underline{v}_{hi} + O(a^3) \,, \tag{3.8.17}$$

where $\underline{v}_{hi}$ is the unique solution, orthogonal to all buckling modes, of the equation

$$P_{11}[\underline{v},\underline{\eta}] + \frac{1}{2}\, P_{111}[\underline{u}_h,\underline{u}_i,\underline{\eta}] = 0 \,, \tag{3.8.18}$$

and $O(a^3)$ stands for a contribution that tends to zero for $a_h \rightarrow 0$ as a homogeneous polynomial of the third degree in a_h.

Equation (3.8.18) may again be replaced by a similar equation in terms of a completely arbitrary kinematical displacement field $\underline{\zeta}$ which is not subject to any orthogonality restriction. In view of (3.8.11) this equation is again identical in form to (3.8.18)

$$P_{11}[\underline{v},\underline{\zeta}] + \frac{1}{2} P_{111}[\underline{u}_h,\underline{u}_i,\underline{\zeta}] = 0 \ . \tag{3.8.19}$$

The minimum value of (3.8.13) may now be written in either of the forms

$$P_2[a_h a_i \underline{v}_{hi}] + P_{21}[a_h \underline{u}_h, a_i a_j \underline{v}_{ij}] + O(a^5) = -P_2[a_h a_i \underline{v}_{hi}] + O(a^5) =$$

$$= -C_{hijk} a_h a_i a_j a_k + O(a^5) \ , \tag{3.8.20}$$

where the coefficients C_{hijk} are defined by

$$C_{hijk} = \frac{1}{2} P_{11}[\underline{v}_{hi}, \underline{v}_{jk}] \ . \tag{3.8.21}$$

From the inequality (3.8.5) we now obtain two inequalities similar to (3.6.38) and (3.6.39)

$$P[a_h \underline{u}_h + \underline{v}] \geqq [B_{hijk} - \frac{1}{1-\varepsilon} C_{hijk}] a_h a_i a_j a_k + O(a^5), \tag{3.8.22}$$

$$\underset{(a_h = \text{const.})}{\text{Min}} P[a_h \underline{u}_h + \underline{v}] \leqq [B_{hijk} - \frac{1}{1+\varepsilon} C_{hijk}] a_h a_i a_j a_k + O(a^5), \tag{3.8.23}$$

where ε is again a positive number as small as we please. Introducing the constants

$$A_{hijk} = B_{hijk} - C_{hijk} \ , \tag{3.8.24}$$

we obtain from (3.8.23) the <u>necessary condition for stability of</u>

the critical state of neutral equilibrium with multiple buckling modes

$$A_{hijk}a_h a_i a_j a_k \geqq 0 \qquad (3.8.25)$$

for all values of the amplitudes. Likewise, we obtain from (3.8.22) as a sufficient condition for stability

$$A_{hijk}a_h a_i a_j a_k \text{ is positive definite.} \qquad (3.8.26)$$

No verdict on stability is again obtained, if the quartic form in (3.8.25) and (3.8.26) is semi-definite. A further investigation would be required in order to get a decision. Such an investigation is more complicated than the investigation of the indecisive case for a simple buckling mode because the semi-definite quartic form presumably is zero only for one or more special sets of values for the ratios of the amplitudes of the buckling modes. We shall not pursue this investigation since the indecisive case for multiple buckling modes is indeed a rare exception. In the author's experience it has never been encountered in actual stability problems.

Finally, it will be remembered from our discussion at the end of section 3.6 that the vanishing of the third variation ought to be regarded as an "exceptional" case in a mathematical sense. Hence it is even more "exceptional" in the same sense if all $\frac{1}{6}n(n+1)(n+2)$ coefficients A_{hij} in (3.8.11) vanish. The symmetry properties of the structure and/or its buckling modes may of course again overrule this consideration. It may again

result from such symmetry that (3.8.11) is indeed satisfied. Nevertheless, experience indicates a more frequent occurrence of instability from a violation of (3.8.11) than from (3.6.77) in the case of a simple buckling mode.

3.9 Approximate solution of stability problems.

The analysis in the present chapter is essentially mathematical in character, even if we have employed the terminology of the theory of stability. No approximative physical assumptions of any kind have been made so far. In this connection we may ignore the two examples, which have been discussed merely for the purpose of some illustration of the argument. In view of the absence of any approximation, the analysis is entirely rigorous, except for the omission of proofs of existence of solutions for the minimum problems encountered in sections 3.4 and 3.6. Moreover, the analysis is equally applicable to other problems in the calculus of variations, specified by some functional $P[\underline{u}]$ which takes the place of our energy increase functional (3.1.1).

It will cause no surprise, however, that the actual application of the theory to problems of elastic stability requires far-reaching simplifications. The number of problems which are capable of a rigorous solution is small even in the classical linear theory of elasticity. The limitations of the available analytical tools become even more apparent in the more complicated, inherently nonlinear, theory of elastic stability. Approximations of some type, and in most cases even of several types, are nearly always indispensable in order to obtain concrete answers to the stability question in specific problems. It seems therefore appropriate to end the present chapter with a preliminary discussion of some of the more

important types of approximations employed in the theory of elastic stability. This general discussion will be helpful in our future choice of suitable approximations in particular cases, and it will shield us to some extent from incautious approaches. It cannot be avoided here that parts of the argument are perhaps less meaningful to readers without any previous experience in the theory of elastic stability. It may be some comfort to such readers that we shall repeatedly return to these questions in future chapters, where the details of the analysis will clarify the issues involved.

The first and <u>most basic simplification</u> has usually already been applied before the stability problem itself has even been formulated. <u>It consists of a suitable choice of the simplified energy expression (3.1.1) on which the ensuing investigation of stability is based</u>. The adequacy of the chosen energy expression, or of an equivalent formulation of the problem in terms of equations of motion and stress-strain relations, is often taken for granted. Such complacency is indeed more often than not justified, but some exceptions prove the case for extreme caution. It should for instance always be borne in mind that actual problems of elastic stability concern three-dimensional elastic bodies, no matter whether one or two dimensions of the body are small in comparison with the other dimension(s). The representation of the potential energy of the body by a surface integral or line integral in the case of approximately two-dimensional or

one-dimensional structures always merits a comparison with the basic three-dimensional theory.

A sound physical insight is of course by far the best guide in devising appropriate simplifications of the basic energy expression (3.1.1). Without such insight or intuition it would be virtually hopeless to attempt the solution of any problem in elastic stability. Nevertheless, it should be kept in mind that physical insight is no more than a precipitation of past experience, and it may well become unreliable, if extrapolation from this experience is carried too far. A careful mathematical evaluation of the implications of all simplifying physical assumptions is therefore always desirable, if not actually indispensable. Complete rigour in a mathematical sense cannot usually be achieved here, but every effort should be made to establish the limitations on the validity of the theory as a consequence of the assumptions in question.

The foregoing more or less philosophical digressions may again be illustrated by means of the examples which have been discussed before. The energy expression (3.1.4) is the exact equivalent of Euler's famous formulation and investigation of the stability of thin rods in axial compression [7]. We shall discuss this problem in detail in chapters and . It will appear that Euler's theory is completely justified, provided that the cross-section of the bar is compact. On the other hand, an entirely different mode of instability may become more critical for bars with a thin-walled open cross-section. The phenomenon of buckling of such bars in a combined flexural

and torsional mode had been entirely overlooked until less than 30 years ago.

A second instructive lesson to be gained from the first example concerns the structure of the energy functional. In (3.1.4) it consists of one term which represents the elastic strain energy in bending, and a second term which represents the energy of the external loads. The elastic strain energy due to compression of the bar has been neglected. The second variation of the energy functional (3.2.11) consists again of two terms, the first one representing the positive definite second variation of the elastic strain energy, the second, negative term representing the second variation of the energy due to the external loads. Several misconceptions in more general buckling problems, in particular for flat plates, stem from unwarranted generalizations of the experience with the buckling of bars. First of all, it was conjectured that the second variation of the elastic energy is always positive definite and that the possibility of instability would therefore arise only in problems where the second variation of the energy of the external loads is negative [4]. This misconception has even now not quite disappeared. Remnants of it are still to be found in current textbooks on elastic stability, in spite of the fundamental researches of Trefftz [18,19] and Marquerre [11,12] in the thirties of the present century. This misunderstanding is closely related to a second common erroneous conjecture in the buckling of flat plates, where it is often assumed, explicitly or tacitly, that extensions and shears in

the middle surface of the plate may be neglected in buckling, as long as the deflections remain (infinitely) small. The connection between these misconceptions becomes clear, if the origin of the second variation of the energy of the external loads in (3.2.11) is examined more closely. It stems from the second term in (3.1.4), which is itself a consequence of the assumption that the center line is inextensible. If this assumption is dropped, the second variation of the energy of the loads disappears entirely, and it is replaced by an additional term of equal magnitude in the second variation of the elastic strain energy, due to extension of the center line (cf. section). This alternative formulation is in agreement, as it should be, with our second example for the general case of dead loading in which the second variation of the energy (3.2.17) is entirely due to the elastic strain energy.

As a last example of a simplified energy expression (3.1.1) we discuss the background of (3.1.8) for the elastic body under dead loading. This expression represents an approximation of the energy increase for an elastic body which obeys a particular generalization of Hooke's law for finite deformations. The approximation is here embodied in the quadratic terms in the strain components γ_{ij}. Before the simplification in question, they appear in the form

$$G\{\gamma_{ij}\gamma_{ij} + \frac{\nu}{1-2\nu}(\gamma_{hh})^2\} + \frac{1}{2}\Delta E_{ijhk}\gamma_{ij}\gamma_{hk}\,, \qquad (3.9.1)$$

where all components of the tensor ΔE_{ijhk} are small in modulus

compared with the shear modulus G. The small additional terms in (3.9.1) are equivalent to small changes in the elastic constants of the material, which are not known with great precision anyway. On physical grounds we have therefore little hesitation in omitting the small additional terms. A more mathematical justification is also readily available in this case. Because the first term in (3.9.1) is positive definite it is indeed ensured that the additional terms never exceed in absolute value a certain small fraction of the principal term. This argument would fail, however, if the first term were not definite. In such a case the neglection of the additional terms would lack an adequate mathematical substantiation. Even though the coefficients ΔE_{ijhk} are all small compared with the coefficients in the first term, in the case of an indefinite or semi-definite first term, the additional terms might well be comparable with or even dominant over the first term for some choice of the strain components. <u>It is indeed always sound practice in the discussion of "small" terms in our energy expressions to compare them with a positive definite aggregate of similar terms in the same variables</u>, whenever such a comparison is at all possible.

Once the basic simplified energy expression (3.1.1) has been firmly established, the next step in the investigation of stability deals with the examination of the second variation $P_2[\underline{u}]$. A further simplification of this second variation is again desirable in most cases, and it may indeed often be obtained without detrimental effects on the accuracy of the predictions

on stability. The need of proper care, however, can hardly be overemphasized. Physical intuition is less effective here than in the establishment of a suitable simplified complete energy expression (3.1.1).

As an example, we may again discuss the second variation (3.2.17) of the energy in the case of the three-dimensional elastic body under dead loading. All terms in the integrand depend on the displacement gradients $u_{i,j}$ as arguments. The coefficients in the first term S_{ij} are certainly all small compared with the shear modulus G appearing as a factor of the second term; the order of magnitude of S_{ij}/G is at most 10^{-2} to 10^{-3} for most engineering materials in the elastic range. The entire stability problem would be obliterated, however, if the first term in (3.2.17) were unsuspectingly omitted. The remaining expression would be positive definite, and no possibility of instability would arise at all. It is evidently impermissible here to neglect the supposedly small first term. The reason is of course that the second term in the integrand is <u>not a positive definite function of the displacement gradients</u>. Omission of the first term can therefore not be supported by a mathematical argument.

This becomes even more clear, if we introduce in addition to the (fictitious) linear strain tensor, defined in section (3.5)

$$\theta_{ij} = \frac{1}{2}(u_{i,j}+u_{j,i}) \ , \qquad (3.9.2)$$

the (fictitious) linear rotation tensor

$$\alpha_{ij} = \frac{1}{2}(u_{j,i}-u_{i,j}) \ . \tag{3.9.3}$$

The latter skew-symmetric tensor is again fictitious as a rotation tensor in the case of finite displacement gradients; it coincides with the rotation tensor only in the case of an infinitesimal displacement field $\underline{u}$. All displacement gradients may now be expressed in the linear strain and rotation tensors

$$u_{i,j} = \Theta_{ij}-\alpha_{ij} \ , \tag{3.9.4}$$

and the second variation (3.2.17) may be written in the form

$$P_2[\underline{u}] = \int_V [\frac{1}{2} S_{ij}(\Theta_{ih}+\alpha_{ih})(\Theta_{jh}+\alpha_{jh}) + G\{\Theta_{ij}\Theta_{ij} + \frac{\nu}{1-2\nu} (\Theta_{hh})^2\}]dv. \tag{3.9.5}$$

It is now perfectly clear that the first term in (3.9.5) cannot be omitted entirely in comparison with the second term. It also depends, in fact, on the rotation tensor (3.9.3) which does not appear in the second term. But it is also evident from (3.9.5) that the part

$$\frac{1}{2} S_{ij}\Theta_{ih}\Theta_{jh} \tag{3.9.6}$$

of the first term is certainly negligible in comparison with the positive definite second term in the linear strain components. Moreover, in view of the smallness in absolute value of S_{ij}/G, it may also be argued that the average values of α_{ij}, taken in a mean square sense, must be large in comparison with the average

values of Θ_{ij} in a similar sense, if the possibility of instability is to arise at all. The order of magnitude of this ratio must be $(G/S)^{\frac{1}{2}}$, where S is for instance the largest principal stress in absolute value. A simplified version of (3.9.5) is then

$$P_2[\underline{u}] = \int_V [\tfrac{1}{2} S_{ij}\alpha_{ih}\alpha_{jh} + G\{\Theta_{ij}\Theta_{ij} + \frac{\nu}{1-2\nu} (\Theta_{hh})^2\}]dv \; , \tag{3.9.7}$$

and the relative error involved appears to be of the order of magnitude $(S/G)^{\frac{1}{2}}$. The actual error of (3.9.6) in buckling problems of slender structures may be even less because the neglected terms

$$\tfrac{1}{2} S_{ij}(\alpha_{ih}\Theta_{jh} + \Theta_{ih}\alpha_{jh}) = S_{ij}\alpha_{ih}\Theta_{jh} \tag{3.9.8}$$

are usually of the same order of magnitude as the terms (3.9.6). This statement will be substantiated in section 6. It is based on the physical fact that in a buckling mode the linear strain components Θ_{jh} which appear in (3.9.8) with a non-vanishing factor $S_{ij}\alpha_{ih}$ are small compared with the linear strain components which dominate in the second term of (3.9.7) and determine its order of magnitude. The actual error of (3.9.7), in comparison with the original expression (3.2.17), is therefore mostly of order of magnitude S/G times the second term.*)

---

*) The foregoing discussion which resulted in the modified form (3.9.7) of the second variation should not be taken to imply that the latter form would always be the more convenient one for practical applications. The original expression (3.2.17) is often more appropriate, for example in the discussion of neutral equilibrium in section 3.5.

We shall now assume that we have obtained an expression for the second variation of the energy in the simplest possible form for the investigation of the stability problem at hand. Even then it will usually not be possible to discuss this second variation by means of a rigorous mathematical analysis. In most cases we shall have to resort to approximate methods, either in a direct discussion of the second variation or in the solution of the associated linear and homogeneous differential equations. We shall not enter here into a comparison of the numerous available methods. Apart from a few words of caution with respect to further simplification of the differential equations, we shall be content with a brief discussion of the well-known Rayleigh-Ritz method for an approximate direct solution.

The eigenvalue problem in terms of the differential equations and boundary conditions equivalent to the variational equation (3.4.11), and the similar equations of neutral equilibrium equivalent to (3.5.3), often invite to a further simplification by the neglection of certain "small" terms. Particular vigilance is needed before it is even considered to yield to such a temptation. It is admittedly sometimes justified to neglect certain "small" terms in these equations. In most cases, however, this is due to the fact that the second variation of the energy has not yet been reduced to its simplest appropriate form "Small" terms in the differential equations whose omission cannot be justified by a further permissible reduction of the second

variation of the energy should always be regarded with great suspicion. It should always be kept in mind that a differential equation requires the vanishing of a sum of terms, and a supposedly small additional term may have an appreciable effect on the solution. The discussion of small terms in the second variation of the energy is on much safer ground, if it can be based on a comparison with a positive definite group of similar terms. Nevertheless, it can often not be avoided, in the interest of further progress toward an approximate solution, to neglect some "small" terms in the differential equations without adequate prior justification. In such cases it should always be attempted to obtain a justification a posteriori. This may sometimes be done simply by a careful examination of the extent to which the approximate solution actually violates the original equations. In other cases the required justification may be obtained from the theory of asymptotic integration of differential equations containing a small parameter.

By far the most important tool for the approximate solution of problems in elastic stability is provided by <u>the direct Rayleigh-Ritz method of the calculus of variations</u>. This method was originally devised by Rayleigh [14] for the approximate solution of the problem of the fundamental mode of free vibrations. It was already applied with conspicuous success by Timoshenko to the solution of a wide variety of stability problems,*) long before the energy theory of elastic stability was put on a firm foundation by Trefftz [18,19].

*) The reader is referred to Timoshenko's treatise on elastic stability [17] for detailed references to his numerous papers.

<u>The Rayleigh-Ritz method for the investigation of the second variation</u> is based on the minimum problem (3.4.2) or (3.4.3). In the latter form it is, for an appropriate choice of $T_2^*[\underline{u}]$, even identical to Rayleigh's original formulation. The minimum property of the (normalized) minimizing displacement field $\underline{u}_1$ is characterized by (cf. section 3.4)

$$= \frac{P_2[\underline{u}_1+\varepsilon\underline{\zeta}]}{T_2[\underline{u}_1+\varepsilon\underline{\zeta}]} = \frac{P_2[\underline{u}_1]}{T_2[\underline{u}_1]} + O(\varepsilon^2) = \omega_1 + O(\varepsilon^2) \; , \qquad (3.9.9)$$

where $\underline{\zeta}$ is an arbitrary kinematically admissible displacement field, and $O(\varepsilon^2)$ stands for a <u>positive</u> term which tends to zero as ε^2, if ε approaches zero. The Rayleight-Ritz method now consists of the evaluation of

$$\omega_1' = \frac{P_2[\underline{u}]}{T_2[\underline{u}]} \qquad (3.9.10)$$

for some suitably selected kinematically admissible displacement field, which is intended to represent a reasonable approximation to the minimizing displacement field $\underline{u}_1$. The result of the evaluation of (3.9.10) is taken to provide an approximation of the minimum ω_1.

The extraordinary power of the Rayleigh-Ritz method lies in the minimum property expressed by (3.9.9). It implies that a first-order error in the guess for the minimizing displacement field results in a <u>second-order error in</u> ω_1'. Moreover, more severe local errors in the guessed displacement field are evened out by the integrations involved in both numerator and denominator of (3.9.10).

It follows immediately from (3.9.9) that the approximate value ω_1' always exceeds the actual minimum ω_1. The approximate verdict on stability obtained by the Rayleigh-Ritz method always overestimates the stability. If equilibrium is unstable according to the approximate verdict, that is if ω_1' is negative, this equilibrium is certainly actually unstable. On the other hand, a positive value of ω_1' is no guarantee for actual stability.

It needs hardly to be emphasized that a skillful choice for the displacement field $\underline{u}$ in (3.9.10) is of paramount importance for the achievement of an accurate prediction on stability. Considerable scope exists here again for the deployment of past experience and physical insight. Some assistance may also be obtained from the differential equations and dynamic boundary conditions, even if they have defeated attempts at a complete rigorous solution. In particular it is often not too difficult to choose the displacement field $\underline{u}$ in (3.9.10) in such a way that it satisfies the dynamic boundary conditions, in addition to the geometric boundary conditions. The latter conditions must of course always be satisfied by a kinematically admissible displacement field.

The chances of a satisfactory result are furthermore greatly enhanced, if we incorporate in the displacement field $\underline{u}$ a number of suitably selected indeterminate parameters c_k ($k=1,2,\ldots$). Upon substitution of the displacement fields $\underline{u}(c_k)$ into (3.9.10), the best possible approximation is obtained as the solution of the algebraic minimum problem

$$\omega_1' = \text{Min.}\ \frac{P_2[\underline{u}(c_k)]}{T_2[\underline{u}(c_k)]}\ . \tag{3.9.11}$$

The expression to be minimized is here a function of the parameters c_k. The labour involved increases very rapidly with the number of parameters, and a good physical judgment is again invaluable, if satisfactory results are to be obtained without excessive effort. In Ritz's special form of the method the parameters c_k all enter into the displacement field $\underline{u}(c_k)$ in a linear manner. This field is then described by

$$\underline{u} = \sum_k c_k\, \underline{u}^{(k)}\ , \tag{3.9.12}$$

where each of the <u>chosen</u> displacement fields $\underline{u}^{(k)}$ (which should not be confused with the eigenfunctions) satisfies the geometric boundary conditions. This form of the method has the advantage that both numerator and denominator in (3.9.11) are homogeneous quadratic functions in the parameters c_k. The resulting conditions for a minimum of (3.9.11) are then homogeneous and linear algebraic equations, for which many effective methods of solution are available. Nevertheless, it is often more convenient to include one or more parameters in $\underline{u}(c_k)$ in a nonlinear form, for example as an indeterminate wave length, in order to achieve an accurate approximation of the minimizing displacement field $\underline{u}_1$ with the smallest possible number of indeterminate parameters.

The Rayleigh-Ritz method has a serious drawback in that it furnishes only an upper bound for the minimum ω_1. In order that the accuracy of an approximation may be established with certainty,

a lower bound for ω_1 is also required. Such a lower bound is even more important than an upper bound from the point of view of safety in engineering applications because it permits a conservative prediction on the stability of a structure. Unfortunately, it is a far more difficult problem to establish a lower bound for a minimum ω_1 with some accuracy [8,16]. We shall not discuss any of the methods devised for this purpose because their application is hardly worthwhile in most problems. Any available additional effort for the establishment of a lower bound is usually more rewardingly spent on an improvement of previously established upper bounds.

The application of the Rayleigh-Ritz method to the determination of critical loads in the theory of elastic stability is conventionally formulated in a slightly different manner. A critical load is defined here as an external load system under which the fundamental state is a critical case of neutral equilibrium. The semi-definite character of the second variation $P_2[\underline{u}]$, resulting in a solution $\omega_1=0$ of minimum problem (3.4.2), may alternatively be expressed by the statement that the minimum value zero of $P_2[\underline{u}]$ is attained for a non-vanishing displacement field $\underline{u}_1$. In the case of our first example of the <u>bar under end loads</u>, whose second variation of the energy is given by (3.2.11), the critical load N_1 may therefore be characterized by

$$N_1 = \text{Min.} \frac{\int_0^{\ell} \frac{1}{2} Bw''^2 dx}{\int_0^{\ell} \frac{1}{2} w'^2 dx} , \qquad (3.9.13)$$

with geometric boundary conditions $w(0)=w(\ell)=0$.

A similar formulation is often given for <u>the general three-dimensional elastic body under dead loading</u>, whose second variation is given by (3.2.17). The initial stresses S_{ij} are then specified in terms of a unit system of initial stresses $S_{ij}^{(o)}$ and a load factor λ

$$S_{ij} = \lambda S_{ij}^{(o)} \qquad (3.9.14)$$

The condition for a semi-definite second variation of the energy (3.2.17) may then be written in the form

$$\text{Min.}\{\int_V G[\tfrac{1}{4}(u_{i,j}+u_{j,i}) + \frac{\nu}{1-2\nu}(u_{h,h})^2]dv +$$

$$+ \lambda_1 \int_V \tfrac{1}{2} S_{ij}^{(o)} u_{h,i} u_{h,j} dv\} = 0 , \qquad (3.9.15)$$

where $\lambda=\lambda_1$ denotes the critical load factor. The minimum problem (3.9.14) is now conventionally replaced by

$$\lambda_1 = \text{Min.} \frac{\int_V G[\frac{1}{4}(u_{i,j}+u_{j,i})(u_{i,j}+u_{j,i}) + \frac{\nu}{1-2\nu}(u_{h,h})^2]dv}{-\int_V \frac{1}{2} S_{ij}^{(o)} u_{h,i} u_{h,j} dv} . \qquad (3.9.16)$$

Some comments are called for in connection with the traditional formulation of the critical load problem in the form (3.9.16). It rests on a first tacit assumption that

(3.9.14) is an appropriate representation for the initial stresses. This assumption is justified, if and only if the change in geometry of the body in the fundamental state I, compared with the undeformed state, may be neglected (cf. chapter 6). In such a case we have indeed (approximate) proportionality of the initial stresses with the load factor, and the differences in geometry of the body in state I for different values of the load factor may be neglected in the evaluation of the integrals in (3.9.15) and (3.9.16). The case in which the fundamental stateis a state of pure strain (cf. chapter 6) represents an important group of examples in which this assumption is justified. In fact, it is justified in most of the more important problems of elastic stability, but exceptions do exist, and it should always be verified whether the application of (3.9.15) or (3.9.16) is legitimate in any particular case.

A second tacit assumption is made in the transition from (3.9.15) to (3.9.16). It is assumed here that the denominator in (3.9.16) is positive definite, and this assumption is incorrect in a wide class of problems. It often happens that both positive and negative critical load factors exist, in which cases the denominator in (3.9.16) is indefinite. No solution exists of problem (3.9.16) in such cases because the functional in (3.9.16) is unbounded. The obvious remedy here is to consider instead of (3.9.16) the reciprocal maximum and minimum problems

$$\left.\begin{aligned} 1/\lambda_1 &= \text{Max.} \\ 1/\lambda_1^- &= \text{Min.} \end{aligned}\right\rangle \frac{-\int_V \frac{1}{2} S_{ij}^{(o)} u_{h,i} u_{h,j} dv}{\int_V G[\frac{1}{4}(u_{i,j}+u_{j,i})(u_{i,j}+u_{j,i}) + \frac{\nu}{1-2\nu}(u_{h,h})^2] dv} . \tag{3.9.17}$$

If the body is properly supported, the denominator in (3.9.17) is positive for all non-vanishing displacement fields. The maximum value of (3.9.17) is then the reciprocal value of the positive critical load factor λ_1, and the minimum of (3.9.17) is the reciprocal value of the negative critical load factor λ_1^-. Equilibrium in the fundamental state is stable for $\lambda_1^- < \lambda < \lambda_1$.

The Rayleigh-Ritz method may also be applied to the investigation of a critical case of neutral equilibrium. The quantity A_4 (3.6.40), which gives the verdict on stability in the case of a simple buckling mode $\underline{u}_1$, depends on the solution $\underline{v}_2$ of the minimum problem

$$P_2[\underline{v}] + P_{21}[\underline{u}_1, \underline{v}] = \text{Min.} \tag{3.9.18}$$

under the side condition

$$T_{11}[\underline{u}_1, \underline{v}] = 0 . \tag{3.9.19}$$

If we assume a displacement field $\underline{v}$ which satisfies (3.9.19), and which contains a number of indeterminate parameters, we obtain the best possible approximation $\underline{v}_2'$, if we minimize (3.9.18) with respect to these parameters. In view of the minimum property of the exact solution, we have in

$$A_4' = P_4[\underline{u}_1] - P_2[\underline{v}_2'] \tag{3.9.20}$$

an <u>upper</u> bound for A_4. Here again the Rayleigh-Ritz method overestimates stability.

The foregoing discussion presupposes that the critical case of neutral equilibrium has been established rigorously, i.e. that the exact solution ω_1 of minimum problem (3.4.2) is zero, and that that minimizing displacement field $\underline{u}_1$ is completely known. If we have already previously applied the approximate Rayleigh-Ritz method to establish an approximation for the critical load, we may again apply this method in order to investigate the stability of the (approximate) critical case of neutral equilibrium. The buckling mode employed in this investigation is then also approximate. Hence our investigation of the minimum problem (3.9.18), (3.9.19), where the exact buckling mode has been replaced by its approximation, is also approximate in the sense that even its exact solution would provide no more than an estimate for A_4. Moreover, we then no longer know whether this estimate is too large or too small.

References to chapter 3.

1. C. B. BIEZENO and H. HENCKY. On the general theory of elastic stability. Proc. Kon. Ned. Ak. Wet. Amsterdam 31 (1928), 569; 32 (1929), 444.

2. C. B. BIEZENO und R. GRAMMEL. Technische Dynamik. Springer, Berlin(1939). Zweite Aufl. in 2 Bde. Springer, Berlin, Göttingen, Heidelberg (1953). English translation by M.L. MEYER, 4 vols. under title "Engineering Dynamics." Blackie & Sons, London(1955,1956). In particular ch.1,§5.

3. O. BOLZA. Vorlesungen über Variationsrechnung. B. G. Teubner, Leipzig und Berlin (1909).

4. G. H. BRYAN. On the stability of elastic systems. Proc. Cambr. Phil. Soc. 6 (1888), 199.

5. R. COURANT und D. HILBERT. Methoden der mathematischen Physik, Bd. 1,2. Aufl. Springer, Berlin (1931). Revised English edition. Interscience Publishers, New York and London (1953).

6. R. COURANT und D. HILBERT. Methoden der mathematischen Physik, Bd. 2. Springer, Berlin (1938). Revised English edition. Interscience Publishers, New York and London (1962).

7. L. EULER. Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes. Additamentum I. De curvis elasticis. Lausanne et Génève (1744). German translation by H. LINSENBARTH in "Ostwald's Klassiker der exakten Wissenschaften", 175, Leipzig (1910). English translation of the addidamentum by W.O. OLDFATHER, C.A. ELLIS and D. M. BROWN. Saint Catherine Press, Bruges (1933).

8. S. H. GOULD. Variational methods for eigenvalue problems. University of Toronto Press, Toronto (1957).

9. J. HADAMARD. Leçons sur le calcul des variations, t. 1. A. Herrmann, Paris (1910).

10. W. T. KOITER. Over de stabiliteit van het elastisch evenwicht. Thesis Delft. H. J. Paris, Amsterdam (1945).

11. K. MARGUERRE. Über die Behandlung von Stabilitätsproblemen mit Hilfe der energetischen Methode. Zeitschr. f. angew. Math. u. Mech. 18 (1938), 57.

12. K. MARGUERRE. Über die Anwendung der energetischen Methode auf Stabilitätsprobleme. Jahrbuch 1938 der Deutschen Versuchsaustalt für Luftfahrt, 252.

13. A. PFLÜGER. Stabilitätsprobleme der Elastostatik. Springer, Berlin, Göttingen, Heidelberg (1950).

14. LORD RAYLEIGH. Theory of sound, 2 vols. Macmillan, London (1877/78). Second edition. (1894/96). Reprinted by Dover Publications, New York (1945).

15. R. V. SOUTHWELL. On the general theory of elastic stability. Phil. Trans. Roy. Soc. London A213 (1913), 187.

16. G. TEMPLE and W. G. BICKLEY. Rayleigh's principle and its applications to engineering. Oxford University Press, London (1933).

17. S. P. TIMOSHENKO. Theory of elastic stability. McGraw-Hill, New York (1936). Second Edition in collaboration with J. M. GERE (1961).

18. E. TREFFTZ. Über die Ableitung der Stabilitätskriterien des elastischen Gleichgewichts aus der Elastizitätstheorie endlicher Deformationen. Proc. 3. Int. Congr. Appl. Mech. 3, Stockholm (1930), 44.

19. E. TREFFTZ. Zar Theorie der Stabilität des elastischen Gleichgewichts. Zeitschr. f. angew. Math. u. Mech. 13 (1933), 160.